A **set** is a collection of objects, each of which is called an **element** of the set.

If S is a set then we use the notation "$x \in S$" to mean "x is an element of set S."

If S is a set and T is also a set there are several possible relationships between S and T.

① "S is a **subset** of T" means all elements of S are also elements in T, so $\forall x \in S, x \in T$.

We write "$S \subseteq T$", and it also means the implication $(x \in S) \Rightarrow (x \in T)$ is true.

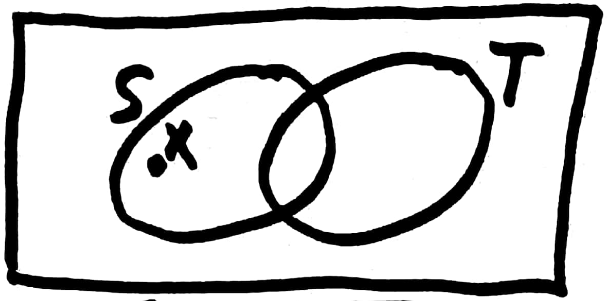
Ex: $\{X \in F^n \mid AX = B\} \subseteq F^n$

Def. The "empty set" = "$\emptyset$" is the set with **no elements**, so $x \in \emptyset$ is false for any choice of x. **Note:** $\emptyset \subseteq S$ is true for any set S because "if $x \in \emptyset$ then $x \in S$" is true since $x \in \emptyset$ is false for any x.

Notation: "$x \notin S$" means "not$(x \in S)$".

not$(S \subseteq T)$ means not$(x \in S \Rightarrow x \in T)$ which is true when $(x \in S \Rightarrow x \in T)$ is false, that is, when $x \in S$ and $x \notin T$ for some x.

Recall "Venn diagrams": They help to visualize set relationships.



$\exists x \in S, x \notin T$

In basic logic there are some statements which are always true for any choices of truth value of the "variables". These are called "tautologies" 50

<u>Ex</u>: $(P \Rightarrow Q) \Leftrightarrow (\text{not}(Q) \Rightarrow \text{not}(P))$

<u>Pf</u>. Use tables of truth values:

P	Q	$P \Rightarrow Q$	not(Q)	not(P)	not(Q) $\Rightarrow$ not(P)
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

same values

EX: not(not P) $\Leftrightarrow P$

Say not(Q) $\Rightarrow$ not(P) is the <u>contrapositive</u> of $P \Rightarrow Q$.

Say $Q \Rightarrow P$ is the <u>converse</u> of $P \Rightarrow Q$.

But $P \Rightarrow Q$ is <u>not</u> <u>equivalent</u> to $Q \Rightarrow P$, that is, $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$ is <u>FALSE</u>.

<u>Tables</u>:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

not always same

<u>NOTE</u>:

The most common logic error made by students is to confuse implication $P \Rightarrow Q$ with its converse $Q \Rightarrow P$. <u>Watch out for this error</u>!

In set theory we often use these constructions of new sets from given sets.

<u>Intersection</u>: $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$

<u>Union</u>: $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$

<u>Complement</u> (difference):

$$S \setminus T = \{x \in S \mid x \notin T\}$$

Basic set notation: $\{x \in S \mid P(x)\}$ where $P(x)$ is an assertion about x, a condition required of x for membership in the set but only $x \in S$ are considered.

Final comments on quantifiers:

When more than one quantifier occurs in an expression, some care about their order must be taken. Let $P(x,y)$ be an assertion about elements x and y.

$$(\forall x \in S, \forall y \in T, P(x,y)) \Leftrightarrow (\forall y \in T, \forall x \in S, P(x,y))$$

since they both mean $P(x,y)$ is true for all choices of elements $x \in S$, $y \in T$.

$\forall x \in S, \exists y \in T, P(x,y)$ means for any choice of $x \in S$ there is some $y \in T$, which may depend on x, such that $P(x,y)$ is true.

<u>Ex</u>: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x+y=0$ is true since $y=-x$ in $\mathbb{R}$ works.

But the quantified statement $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x+y=0$ is <u>false</u> since it says there is a $y \in \mathbb{R}$ such that all $x \in \mathbb{R}$ satisfy $x+y=0$. No real number y is the additive inverse of <u>all</u> real numbers x.

So generally you <u>cannot</u> change the order of $\forall\exists$ to $\exists\forall$ (or $\exists\forall$ to $\forall\exists$) without changing the meaning and truth of the expression.

There is no problem with

$$(\exists x \in S, \exists y \in T, P(x,y)) \Leftrightarrow (\exists y \in T, \exists x \in S, P(x,y))$$

since they say the same thing.

Back to linear algebra.

Recall the definition of a linear map $L: F^n \to F^m$ requires that $\forall X, Y \in F^n$, $\forall \alpha \in F$,

① $L(X+Y) = L(X) + L(Y)$, ② $L(\alpha \cdot X) = \alpha \cdot L(X)$.

For such a lin. map L, can we always find a matrix $A = [a_{ij}] \in F^m_n$ such that $\forall X \in F^n$ $L(X) = L_A(X) = AX$?

Let's try!

Step ①: Write $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

where $S = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ is a very special list in F^n.

We call this list S the "standard basis of F^n". So we have $X = \sum_{j=1}^{n} x_j e_j$ for short. 56

Step ②: Use rule ① of linearity for L to get

$$L(X) = L(x_1e_1 + \cdots + x_n e_n) = L(x_1 e_1) + L(x_2 e_2 + \cdots + x_n e_n)$$
$$= L(x_1 e_1) + L(x_2 e_2) + L(x_3 e_3 + \cdots + x_n e_n) = \cdots$$
$$= L(x_1 e_1) + L(x_2 e_2) + \cdots + L(x_n e_n) = \sum_{j=1}^{n} L(x_j e_j)$$

Use rule ② of lin. of L to get

$$L(X) = x_1 L(e_1) + x_2 L(e_2) + \cdots + x_n L(e_n) = \sum_{j=1}^{n} x_j L(e_j).$$

Step ③: For each $1 \le j \le n$ let $L(e_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in F^m$ and let $A = [a_{ij}] \in \mathbb{F}_n^m$.

We have thus used L to get a matrix $A \in F_n^m$ such that $Col_j(A) = L(e_j)$ and so $L(X) = \sum_{j=1}^{n} x_j \, Col_j(A) = AX = L_A(X)$. (see p.10)

<u>Th.</u> If $L : F^n \to F^m$ is a lin. map then $L = L_A$ for the matrix $A \in F_n^m$ uniquely determined by L such that $Col_j(A) = L(e_j)$ for $1 \le j \le n$.

<u>Ex</u>: $L : F^3 \to F^2$ defined by $L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2 + 5x_3 \\ 2x_1 \qquad + x_3 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 & 5 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ since $L(e_1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $L(e_2) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $L(e_3) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

<u>Th</u>. Let $L = L_A : F^n \to F^m$ for $A \in F^m_n$.
L is <u>injective</u> iff $Ker(L) = \{0^n_1\}$.

<u>Pf</u>. As an "iff" there are two parts to the proof, "$\Rightarrow$" and "$\Leftarrow$".

First suppose L is injective. Try to prove that $Ker(L) = \{0^n_1\}$ to get "$\Rightarrow$" part.

L inj. means $\forall X, Y \in F^n$, $L(X) = L(Y) \Rightarrow X = Y$.

From p.12, $L(0^n_1) = A0^n_1 = 0^m_1$ so $0^n_1 \in Ker(L)$ is true for any linear map L.

If $X \in Ker(L)$ then $L(X) = 0^m_1 = L(0^n_1)$ so inj. of L gives $X = 0^n_1$ which means $Ker(L) = \{0^n_1\}$.

Second, suppose $Ker(L) = \{0^n_1\}$. Try to show L inj.

To show L inj. we must prove the "implies" statement: "Suppose $X, Y \in F^n$ and $L(X) = L(Y)$. Show that $X = Y$." From $L(X) = L(Y)$ we know $L(X) - L(Y) = 0_1^m$ (using basic algebra rules for matrices), so $L(X) + (-1)L(Y) = 0_1^m$ (def. of $-$) $= L(X) + L(-Y) = L(X - Y)$ (by lin. of L). Thus $L(X - Y) = 0_1^m$ which means $X - Y \in \ker(L) = \{0_1^n\}$ so $X - Y = 0_1^n$ giving $X = Y$. $\square$ (end of proof)

This means that to check when L is inj. we solve $L(X) = 0_1^m$. L is inj. exactly when the <u>only</u> solution is $X = 0_1^n$. For $L = L_A$ it means $[A|0_1^m] \xrightarrow{r.r.} [C|0_1^m]$ with $\mathrm{rank}(C) = n$.

Th. $L_A : F^n \to F^m$, for $A \in F^m_n$, is inj. iff rank$(A) = n$. 160

Th. $L_A : F^n \to F^m$, for $A \in F^m_n$, is surj. iff rank$(A) = m$.

Pf. L_A is surj. iff Range$(L_A) = F^m$

iff $\{B \in F^m \mid AX = B$ is consistent$\} = F^m$

iff $AX = B$ is consistent for all $B \in F^m$

iff $[A|B] \xrightarrow{r.r.} \underset{\text{RREF}}{[C|D]}$ and rank$(C) = m$ (no zero rows in C). $\square$

So rank(A) tells us whether or not L_A is surj., inj., bijective, invertible.

Examples: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \in F_4^3$ so $L_A : F^4 \to F^3$

$A \underset{row}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ in RREF has rank $r = 1$ so

L_A is not inj since $r = 1 < 4 = n$

and $\mathrm{Ker}(L_A) = \{X \in F^4 \mid AX = 0\}$ has $n - r = 4 - 1 = 3$ free var's

Also $AX = B$ is not always consistent,

$[A|B] \xrightarrow{r.r.} \left[\underbrace{\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}}_{C} \middle| D \right]$ $D = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 3b_1 \end{bmatrix}$ $0 = b_2 - 2b_1$, $0 = b_3 - 3b_1$

so $b_2 = 2b_1$, $b_3 = 3b_1$, $b_1 \in F$ free

L_A is <u>not surj.</u> since $r = 1 < 3 = m$.

$$\mathrm{Range}(L_A) = \left\{ \begin{bmatrix} b_1 \\ 2b_1 \\ 3b_1 \end{bmatrix} \in F^3 \middle| b_1 \in F \right\}.$$