<u>Th.</u> For $A \in F^n_n$, A is invertible iff $A \underset{\text{row}}{\sim} I_n$ iff $\text{rank}(A) = n$. In that case, to find A^{-1}; row reduce $[A|I_n] \xrightarrow{\text{r.r.}} [I_n|A^{-1}]$.

<u>Pf.</u> We have seen that when $A \underset{\text{row}}{\sim} I_n$ and $[A|I_n] \xrightarrow{\text{r.r.}} [I_n|B]$ then B satisfies $AB = I_n$. We need to also show that $BA = I_n$. Each step (elem. row op.) of the row reduction can be reversed, so $[I_n|B] \xrightarrow{\text{r.r.}} [A|I_n]$. If we do those steps to $[B|I_n]$ then we get $[I_n|A]$ so $[B|I_n] \xrightarrow{\text{r.r.}} [I_n|A]$ which means $BA = I_n$. $\square$

Ex: Is $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -5 \\ 1 & 3 & -7 \end{bmatrix}$ invertible? Try to find inverse by row reducing

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 3 & -5 & 0 & 1 & 0 \\ 1 & 3 & -7 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 2 & -6 & -1 & 0 & 1 \end{array}\right] \to$$

$+ \quad +$

$-2 \ -2 \ 2 \ -2 \ 0 \ 0$ $\qquad$ $0 \ -2 \ 6 \ 4 \ -2 \ 0$

$-1 \ -1 \ 1 \ -1 \ 0 \ 0$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 & -2 & 1 \end{array}\right]$$

But the row of zeros on the last row of left side stands for the equations in the 3 lin. systems

$0x_1 + 0x_2 + 0x_3 = 3$ or -2 or 1 needing to be solved. $0 \neq 3$ so system is <u>not</u> consistent, no solutions, A is <u>not</u> invertible.

What can you say about $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ for this A?

Def. Matrix $E \in F_n^n$ obtained from I_n by an elem. row op. done to I_n is called the elem. matrix associated with that elem. row op.

Examples: ① $E_s = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is the elem. matrix associated with the switcher row op. $R_1 \leftrightarrow R_3$.

② $E_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the elem. matrix associated with the multiplier row op. $cR_2 \to R_2$.

③ $E_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$ is the elem. matrix associated with the adder row op. $cR_1 + R_3 \to R_3$.

Note: $E_s^{-1} = E_s$, $E_M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}$.

Th: Let $E \in F^m_m$ be the elem. matrix associated with an elem. row op. Then $\forall A \in F^m_n$, $B = EA$ is the matrix obtained from A by doing that row op. to A, so $EA \underset{row}{\sim} A$.

Ex: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ is A after switching rows 1 and 2.

$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4c & 5c & 6c \end{bmatrix}$ is A after multiplying row 2 by c.

$\underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{E = E_{Adder}}\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{A} = \overset{B}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}}$. Continuing to row reduce B we have

$E_M B = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = C$ and then

$E'_{Adder} C = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = D$ is in RREF.

So we found a finite sequence of elem. matrices, $E_1 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

such that $E_3 E_2 E_1 A = D$ in RREF. The row reduction of A to D was achieved by the left multiplication of elem. matrices corresponding to the elem. row ops.

__Th.__ For $A, B \in F^m_n$, $A \underset{\text{row}}{\sim} B$ iff $B = E_t \cdots E_2 E_1 A$ for some elem. matrices $E_1, \dots, E_t \in F^m_m$.

__Th.__ For $A \in F^n_n$, A is invertible iff $A \underset{\text{row}}{\sim} I_n$ iff A is a product of elem. matrices.

Pf. We have seen that A is invertible **B1**
iff $[A|I_n] \xrightarrow{\text{r.r.}} [I_n|B]$ so that $AB = I_n$ and
$A \underset{\text{row}}{\sim} I_n$. The row reduction process on both sides of $[A|I_n]$ can be achieved using elem. matrices, $E_1, \dots, E_t$,

$$[A|I_n] \to [E_1A|E_1I_n] \to [E_2E_1A|E_2E_1I_n] \to \cdots$$

so finally $[E_t \cdots E_1 A | E_t \cdots E_1 I_n] = [I_n | B = A^{-1}]$.
From the right side get $E_t \cdots E_1 = B = A^{-1}$
and from the left get $(E_t \cdots E_1)A = I_n$.
Each elem. matrix is invertible and its inverse is also elem. (of same type), so
$A = E_1^{-1}E_2^{-1} \cdots E_t^{-1}$ is a product of elem. matrices.

$\square$

Cor. For $A, B \in F^m_n$, $A \underset{\text{row}}{\sim} B$ iff $B = QA$ for some invertible $Q \in F^m_m$.

Th: Let $E \in F^n_n$ be the elem. matrix obtained from I_n by doing an elem. **col.** op to I_n. Then $\forall A \in F^m_n$, $B = AE$ is the matrix obtained from A by doing that col. op. to A.

Ex: $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 6 & 3 \end{bmatrix}$ is A after switching col_1 and col_2,

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 5 \\ 9 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -3 \\ 3 & -6 \end{bmatrix}$$

Note: Col. ops. do **not** help to solve lin. systems!

Def: For $A, B \in F^m_n$, say $A \underset{col}{\sim} B$ (column equivalent) when B can be obtained from A by a finite sequence of elem. col. ops.

Th: $\underset{col}{\sim}$ is an equiv. relation on F^m_n (symm, reflex, trans.) and $A \underset{col}{\sim} B$ iff $B = AP$ for some invertible $P \in F^n_n$ (which is a product of elem. matrices).

Th. Each $A \in F^m_n$ is $\underset{col}{\sim}$ to a unique matrix in Reduced Column Echelon Form (RCEF).

Def. For $A, B \in F^m_n$ say $A \underset{row/col}{\sim} B$ (row/col equivalent) when $B = QAP$ for some invertible matrices $P \in F^n_n$ and $Q \in F^m_m$.

What is the "nicest" B s.t. $A \underset{row/col}{\sim} B$?

Th. If $A, B \in F_n^n$ are invertible then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

Pf. To check that $(AB)^{-1} = B^{-1}A^{-1}$ we only need to verify that $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$.

By assoc. law of matrix mult. we have

$$(AB)(B^{-1}A^{-1}) = A(B(B^{-1}A^{-1})) = A((BB^{-1})A^{-1}) =$$

$$A(I_nA^{-1}) = AA^{-1} = I_n \quad \text{and}$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}(AB)) = B^{-1}((A^{-1}A)B) = B^{-1}(I_nB)$$

$$= B^{-1}B = I_n. \quad \square$$

Cor: If $A_1, A_2, \dots, A_t \in F_n^n$ are invertible, so is their product, and $(A_1A_2\cdots A_t)^{-1} = A_t^{-1}\cdots A_2^{-1}A_1^{-1}$.

Ex. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ so

$$A^{-1} = \frac{1}{3-2}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \text{ and } B^{-1} = \frac{1}{0-(-1)}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} \text{ and } (AB)^{-1} = \frac{1}{-2-(-3)}\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} = (AB)^{-1}.$$

Note that

$$A^{-1}B^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix} \neq (AB)^{-1}.$$

Order matters for matrix multiplication!