While matrices are very important and useful, linear algebra applies in more general structures, <u>vector spaces</u>. We will define what it means for V to be a vector space with scalars F by the axioms (algebra laws) obeyed by the operations $+$ and $\cdot$, and then we will prove results true for any vector space, valid in any example. This is how the use of abstraction gives us more power of understanding many examples at once.

Let F be a field (for example, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) with operations $+$ and $\cdot$, and special identity elements 0 and 1, obeying the usual rules.

<u>Def</u>. A set V is called a vector space over F when ① There is an "addition" operation on V, $+ : V \times V \to V$ (closure of V under $+$), that is, $\forall v_1, v_2 \in V$, $\exists\, v = v_1 + v_2 \in V$ obeying the axioms below; ② There is a "scalar multiplication" on V, $\cdot : F \times V \to V$ (closure of V under $\cdot$), that is, $\forall \alpha \in F$, $\forall v \in V$, $\exists\, v' = \alpha \cdot v \in V$ obeying axioms below;

<u>Axioms</u>: ③ $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ (assoc. of $+$ in V)

④ $v_1 + v_2 = v_2 + v_1$ (commutativity of $+$ in V)

⑤ $\exists\, \theta = \theta_V \in V$, $\forall v \in V$, $v + \theta = v$ (Iden. elt. for $+$)

⑥ $\forall v \in V$, $\exists -v \in V$ s.t. $v + (-v) = \theta$ (Additive inverse)

⑦ $\forall \alpha \in F$, $\forall v_1, v_2 \in V$, $\alpha \cdot (v_1 + v_2) = (\alpha \cdot v_1) + (\alpha \cdot v_2)$

⑧ $\forall \alpha_1, \alpha_2 \in F$, $\forall v \in V$, $(\alpha_1 + \alpha_2) \cdot v = (\alpha_1 \cdot v) + (\alpha_2 \cdot v)$

⑨ $\forall \alpha_1, \alpha_2 \in F$, $\forall v \in V$, $\alpha_1 \cdot (\alpha_2 \cdot v) = (\alpha_1 \cdot \alpha_2) \cdot v$

⑩ $\forall v \in V$, $1 \cdot v = v$.

<u>Th</u>. For $1 \leq m, n \in \mathbb{Z}$, the set of all $m \times n$ matrices with entries from field F, F_n^m, is a vector space over F with $+$ as matrix addition, $\cdot$ as scalar multiplication, and "zero vector" $\theta = O_n^m$, the $m \times n$ matrix with all entries $0 \in F$.

<u>Pf</u>. Exercise using just the definitions of $+$ and $\cdot$ for F_n^m. $\square$

Example. How do you prove axiom ③ in F_n^m? 89

Let $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[c_{ij}] \in F_n^m$. Then

$(A+B)+C = [(a_{ij}+b_{ij})]+[c_{ij}] = [(a_{ij}+b_{ij})+c_{ij}]$ and

$A+(B+C) = [a_{ij}]+[(b_{ij}+c_{ij})] = [a_{ij}+(b_{ij}+c_{ij})]$.

Because + in field F is associative, for each $1 \le i \le m$, $1 \le j \le n$, we have

$$(a_{ij}+b_{ij})+c_{ij} = a_{ij}+(b_{ij}+c_{ij}) \in F,$$ so

the (i,j)-entries of $(A+B)+C$ and $A+(B+C)$ are equal, so the matrices are equal,

$$(A+B)+C = A+(B+C).$$

There are many more examples of vector spaces besides the matrix examples.

<u>Polynomials:</u> Given a fixed $0 \le m \in \mathbb{Z}$, let

$$\mathcal{P}_m[t] = \{a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m \mid a_i \in F \text{ for } 0 \le i \le m\}$$

be the set of all polynomials in variable t of degree $\le m$ with coefficients from F. We write more briefly, $p(t) = \sum_{i=0}^{m} a_i t^i$ for an element of $\mathcal{P}_m[t]$. Addition and scalar mult. are defined as usual by

$$\left(\sum_{i=0}^{m} a_i t^i\right) + \left(\sum_{i=0}^{m} b_i t^i\right) = \sum_{i=0}^{m} (a_i + b_i) t^i \quad \text{and}$$

$$p(t) + q(t) = (p+q)(t)$$

$$\alpha \cdot p(t) = \alpha \cdot \left(\sum_{i=0}^{m} a_i t^i \right) = \sum_{i=0}^{m} (\alpha \cdot a_i) t^i.$$

The "zero polynomial" is $0 = \sum_{i=0}^{m} 0 t^i$, the poly. with all zero coefficients.

<u>Th</u>: $P_m[t]$ with coefficients from F is a vector space over F.

<u>Def</u>. For field F, let $P[t] = \left\{ \sum_{i=0}^{m} a_i t^i \;\middle|\; \begin{array}{l} 0 \le m \\ a_i \in F \end{array} \right\}$ be the set of all polynomials in t with no restriction on the degree of $p(t) \in P[t]$.

<u>Note</u>: $\deg(p(t)) = m$ when $0 \neq a_m$ is the coeff. of the highest power t^m in $p(t)$.

Th: $\mathcal{P}[t]$ is a vector space over F.

Ex. (Function spaces).

Let $Fun(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define $+$ and $\cdot$ by: $\forall f, g \in Fun(\mathbb{R})$, let $f+g : \mathbb{R} \to \mathbb{R}$ be the function s.t. $\forall x \in \mathbb{R}$, $(f+g)(x) = f(x) + g(x)$ and $\forall \alpha \in \mathbb{R}$, let $(\alpha \cdot f) : \mathbb{R} \to \mathbb{R}$ be the function s.t. $\forall x \in \mathbb{R}$, $(\alpha \cdot f)(x) = \alpha \cdot (f(x))$.

The "zero function" $\theta : \mathbb{R} \to \mathbb{R}$ is $\theta(x) = 0$, $\forall x \in \mathbb{R}$.

Th: $Fun(\mathbb{R})$ is a vector space over $\mathbb{R}$.

Def. Suppose V is a vector space over F. Let $W \subseteq V$ be a subset of V. We say W is a subspace of V when W is itself a vector space with the same $+$ and $\cdot$ as V.

Notation: Write $W \leq V$ when W is a subspace of V.

Th: $W \leq V$ when W satisfies

① $\forall w_1, w_2 \in W$, $w_1 + w_2 \in W$ (Closure of W under $+$)

② $\forall \alpha \in F, \forall w \in W$, $\alpha \cdot w \in W$ (Closure of W under $\cdot$)

③ $\theta_V \in W$ (Zero vector of V is in W)

Pf. In order for W to be a vector space it must satisfy axioms 1 - 10 on p. 87-88. 94

1 and 2 in this theorem are 1 and 2 for W. **Note**: These are saying **more** than just $w_1 + w_2 \in V$ and $\alpha \cdot w \in V$, which are **already given** since $W \subseteq V$.

Axioms 3, 4, 7, 8, 9, 10 are true for all elements of V so they are true for all elements of W since $W \subseteq V$.

Axiom 5 is being assumed for W by 3 above.

Why is 6 true? We need the next Lemma to finish this proof.

Lemma. Let V be any vector space over F. $\lfloor 95$

Then (a) $\forall v \in V$, $0 \cdot v = \theta$.

(b) $\forall \alpha \in F$, $\alpha \cdot \theta = \theta$.

(c) $\forall v \in V$, $(-1) \cdot v = -v$

(d) $\forall \alpha \in F$, $\forall v \in V$, if $\alpha \cdot v = \theta$ then either $\alpha = 0$ or $v = \theta$.

Pf. (a) Let $0 \cdot v = w \in V$. Since $0 = 0 + 0 \in F$,

$w = 0 \cdot v = (0+0) \cdot v = (0 \cdot v) + (0 \cdot v) = w + w$.

By axiom 6, $\exists -w \in V$ s.t. $w + (-w) = \theta$ so

$\theta = w + (-w) = (w+w) + (-w) \overset{3}{=} w + (w + (-w))$

$\overset{5}{=} w + \theta \overset{}{=} w$ so $\theta = 0 \cdot v$.

(b) By axiom ⑤, $\theta + \theta = \theta$, so $\forall \alpha \in F$, if $w = \alpha \cdot \theta$ then $w = \alpha \cdot \theta = \alpha \cdot (\theta + \theta) \overset{⑦}{=} (\alpha \cdot \theta) + (\alpha \cdot \theta) = w + w$ and by the same argument as in (a), $w = w + w$ gives $w = \theta$

so $\alpha \cdot \theta = \theta$.

(c) In axiom ⑥, $\forall v \in V$, $\exists -v \in V$ s.t. $v + (-v) = \theta$, there is a prejudice hidden in the notation $-v$ that this "additive inverse of v" is uniquely determined by v. Let us first check if this is true using the given axioms. Suppose there were two candidates u and w such that $v + u = \theta$ and $v + w = \theta$.

Why should $u = w$ have to be true?

We have by ④ $u+v=v+u=\theta$ so by ③, get [9]

$$w=\theta+w=(v+u)+w=(u+v)+w \overset{③}{=} u+(v+w)$$
$$=u+\theta=u$$

So there is only one unique vector $-v$ determined by v such that $v+(-v)=\theta$, the additive inverse (negative) of $v\in V$.

Back to proving that $-v=(-1)\cdot v$.

In field F, the mult. iden. element 1 has a unique negative, $-1\in F$, such that $1+(-1)=0$.

If we show that $v+(-1)\cdot v=\theta$ then by uniqueness of $-v$, we must have $-v=(-1)\cdot v$.

By axiom ⑩, $v=1\cdot v$, so $v+(-1)\cdot v=1\cdot v+(-1)\cdot v$
$=(1+(-1))\cdot v=0\cdot v=\theta$ by ⑧ and part (a).

(d) Suppose $\alpha \cdot v = \theta$. Either $\alpha = 0$ or $\alpha \neq 0$ in F. In the first case we get what we want. Suppose $\alpha \neq 0$, so $\exists \alpha^{-1} \in F$ so $\alpha^{-1} \cdot (\alpha \cdot v) = \alpha^{-1} \cdot \theta = \theta$ by part (b).

By axiom 9, $\alpha^{-1} \cdot (\alpha \cdot v) = (\alpha^{-1} \cdot \alpha) \cdot v = 1 \cdot v \overset{10}{=} v$

so we get $v = \theta$ as we want. □

Going back to finish the proof on p. 94, $\forall w \in W, (-1) \cdot w \in W$ by the assumption 2 (W is closed under $\cdot$) so the Lemma (c) says $-w = (-1) \cdot w \in W$ giving 6. This completes the proof of the "subspace theorem" on p. 93.

Applications and Examples of subspace.

Th: For $A \in F^m_n$, $L_A : F^n \to F^m$, we have $Ker(L_A) \leq F^n$ and $Range(L_A) \leq F^m$.

Pf. Looking back at p. 14 and p. 38, we see that $Ker(L_A) = W = \{X \in F^n \mid AX = 0^m_1\}$ is closed under + and $\cdot$, and contains 0^n_1, so it is a subspace of F^n by the subspace theorem on p. 93. We also have that $Range(L_A)$ satisfies the assumptions of that theorem: If $L_A(X) = AX$, $L_A(Y) = AY \in Range(L_A)$ then

$L_A(X) + L_A(Y) = L_A(X+Y) \in Range(L_A)$ (see p. 36)

If $L_A(X)=AX \in \text{Range}(L_A)$ and $\alpha \in F$ then
$\alpha \cdot L_A(X) = L_A(\alpha \cdot X) \in \text{Range}(L_A)$ (see p. 36).
$L_A(0_1^n) = 0_1^m \in \text{Range}(L_A)$ (see p. 36). $\square$

On page 37 we defined a general linear map $L : F^n \longrightarrow F^m$ by two axioms. later on p. 55–57 we proved that any linear map $L : F^n \longrightarrow F^m$ is equal to an L_A for a specific $A \in F_n^m$ with $\text{col}_j(A) = L(e_j)$, $1 \le j \le n$.
Now we wish to define the more general kind of linear map $L : V \longrightarrow W$ for any two vector spaces V and W (both over the same field F).