Math 304-6 Linear Algebra Spring 2025 Exam 2 Feingold

SHOW ALL NECESSARY WORK FOR EACH PROBLEM

(1) (10 Points) $A = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & -1 & 5 \\ 3 & 1 & 1 & 1 & 2 \\ 4 & 1 & 2 & -1 & 7 \end{bmatrix}$ row reduces to $C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $L_A : \mathbb{F}^5 \to \mathbb{F}^4$ be the linear map $L_A(X) = AX$.

(a) (2 Points) Find a **basis** for $Row(A)$, the row space of A.

(b) (2 Points) Find a **basis** for $Ker(L_A) = Nul(A)$, the kernel of L_A.

(c) (2 Points) Find a **basis** for $Range(L_A) = Col(A)$.

(d) (4 Points) For each free variable in $Ker(L_A)$, find a **dependence relation** among the columns of A and use it to write a **non-pivot** column as a linear combination of **previous pivot** columns.

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(2) (10 Pts) Answer each question **separately**. V and W are vector spaces over $\mathbb{F}$.

(a) Find the elementary matrix E such that for any $A \in \mathbb{F}^3_n$, EA is the matrix obtained from A by multiplying row 3 of A by 7.

(b) If $S = \{v_1, v_2, \ldots, v_n\}$ is an **independent** subset of V, and $L : V \to W$ is **injective**, what is the most you can be sure about $L(S) = \{L(v_1), L(v_2), \ldots, L(v_n)\}$ in W?

(c) If $S = \{v_1, v_2, \ldots, v_m\}$ is **dependent** in V, and linear map $L : V \to W$ is **injective**, what is the most you can be sure about $L(S) = \{L(v_1), \ldots, L(v_m)\}$?

(d) In W let $T = \{w_1, w_2, \ldots, w_{m-1}\}$ and $T' = \{w_1, w_2, \ldots, w_{m-1}, w_m\}$ where the last vector $w_m \in \langle T \rangle$ is a linear combination of the previous vectors. What is the relationship between $\langle T \rangle$ and $\langle T' \rangle$?

(e) If $S = \{v_1, v_2, \ldots, v_n\}$ is **independent** in V and $v \in \langle S \rangle$ and $T = \{v_1, v_2, \ldots, v_n, v\}$, then what is the most you can be sure about $dim(\langle T \rangle)$?

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(3) (10 points) Answer each question separately.

(a) If $L : \mathbb{F}^3_3 \to \mathbb{F}^9$ is **injective**, what is the most you can be sure about L?

(b) If $L : \mathbb{F}^3 \to \mathbb{F}^8$ what are all the possibilities for $dim(Range(L))$?

(c) If $L : \mathbb{F}^7 \to \mathbb{F}^4$ what are all the possibilities for $dim(Ker(L))$?

(d) If $A \in \mathbb{F}^n_n$ and the homogeneous linear system $AX = 0$ has **non-trivial** solutions, then what is the most you can be sure about $\det(A)$?

(e) Let $A \in \mathbb{F}^n_n$ be **invertible** and suppose a **non-zero** $X \in \mathbb{F}^n$ satisfies $AX = \lambda X$ for some $\lambda \in \mathbb{F}$. Why can you be sure that $\lambda \neq 0$?

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(4) (10 Pts) Answer each question **separately**.

(a) (5 Pts) $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \right\}$ is a basis of $\mathbb{F}^3$. For $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ find the coordinate vector $[v]_S$ with respect to S.

(b) (5 Pts) $T = \{t^2, t^2 + t, t^2 + t + 1\}$ is a basis of $\mathcal{P}_2$ (polynomials of degree at most 2). For $p = at^2 + bt + c$ find the coordinate vector $[p]_T$ with respect to T.

(5) (10 Points) Answer each question separately.

(a) Determine whether $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ is invertible, and find A^{-1} if it is.

(b) Let $E \in \mathbb{F}^n_n$ be an elementary matrix corresponding to an elementary **switcher** row operation. What can you say about $\det(E)$?

(c) Suppose $A \in \mathbb{F}^n_n$ has characteristic polynomial $\det(\lambda I_n - A) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r}$ with r distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ in $\mathbb{F}$. What is the most you can say about the sum of all the algebraic multiplicities, $k_1 + k_2 + \cdots + k_r$?

(d) With notation as in part (c), let $A_{\lambda_i} = \{X \in \mathbb{F}^n \mid AX = \lambda_i X\}$ be the λ_i eigenspace of A. What is the **geometric multiplicity** g_i of A?

(e) If $A^T = A^{-1}$ for $A \in \mathbb{F}^n_n$, what is the most you can say about $\det(A)$?

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(6) (10 Points) Let $L : \mathbb{F}^2 \to \mathbb{F}^3$ be given by $L \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a + 3b \\ -a + b \\ 4a - 5b \end{bmatrix}$.

Let $S = \{v_1, v_2\}$ be the standard basis of $\mathbb{F}^2$ and let $T = \{w_1, w_2, w_3\}$ be the standard basis of $\mathbb{F}^3$. Let other ordered bases be

$$S' = \left\{ v_1' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2' = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \quad \text{and } T' = \left\{ w_1' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, w_2' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, w_3' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(a) (2 pts) Find the matrix ${}_T[L]_S$ representing L from S to T.

(b) (3 pts) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' **without using transition matrices**. Do it directly by row reducing $[T'|L(S')]$.

(c) (3 pts) Find the transition matrices ${}_SP_{S'}$, ${}_TQ_{T'}$ and ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$.

(d) (2 pts) Using your answers from parts (a) and (c), explicitly multiply out ${}_{T'}Q_T\ {}_T[L]_S\ {}_SP_{S'}$ and compare it to your answer for ${}_{T'}[L]_{S'}$ from part (b). What was their relationship, and what should it have been?

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(7) (15 Points) Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(a) (5 pts) Find $Char_A(\lambda) = \det(\lambda I_4 - A)$, the **characteristic polynomial** of A.

(b) (2 pts) Find the **eigenvalues** λ_i of A and their **algebraic multiplicities** k_i.

(c) (6 pts) For each eigenvalue λ_i of A, find a **basis** of its eigenspace A_{λ_i}.

(d) (2 pts) Is A diagonalizable? Explain why. If it is, give a diagonal D similar to A.

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(1) (10 Points) (a) (2 Points)

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & -1 & 5 \\ 3 & 1 & 1 & 1 & 2 \\ 4 & 1 & 2 & -1 & 7 \end{bmatrix} \text{ row reduces to } C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ so}$$

$\{[1 \quad 0 \quad 1 \quad 0 \quad 1], [0 \quad 1 \quad -2 \quad 0 \quad 1], [0 \quad 0 \quad 0 \quad 1 \quad -2]\}$ is a basis for $Row(A)$.

(b) (2 Points) A basis for $Ker(L_A) = Nul(A)$ is found by row reducing $[A|\mathbf{0}_1^4]$ to $[C|\mathbf{0}_1^4]$, interpreting the solutions in $\mathbb{F}^5$, and separating the free variables to get two independent vectors which span it:

$$\left\{ \begin{bmatrix} -r-s \\ 2r-s \\ r \\ 2s \\ s \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{F}^5 \;\middle|\; r, s \in \mathbb{F} \right\} \text{ so } \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis for $Ker(L_A)$.

(c) (2 Points) A basis for $Col(A)$ consists of the three pivot columns of A, the columns with leading ones in the RREF C, that is, $\{Col_1(A), Col_2(A), Col_4(A)\}$. Other correct answers can be obtained by linear combinations of those three columns, for example,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(d) (4 Points) Each basis vector in $Ker(L_A)$ gives a dependence relation among the columns of A. The two dependence relations obtained that way are:

$$-1Col_1(A) + 2Col_2(A) + Col_3(A) = \mathbf{0}_1^4$$

and

$$-1Col_1(A) - 1Col_2(A) + 2Col_4(A) + Col_5(A) = \mathbf{0}_1^4$$

so the two **non-pivot** columns are the linear combinations of **previous pivot** columns

$$Col_3(A) = Col_1(A) - 2Col_2(A) \quad \text{and} \quad Col_5(A) = Col_1(A) + Col_2(A) - 2Col_4(A).$$

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(2) (10 Points, 2 points each)

(a) For elementary matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, obtained by doing the row operation to I_3, EA is the matrix obtained from A by multiplying row 3 of A by 7.

(b) If $S = \{v_1, v_2, \ldots, v_n\}$ is **independent** in V, and $L : V \to W$ is **injective**, then you be sure that $L(S) = \{L(v_1), L(v_2), \ldots, L(v_n)\}$ is an **independent** subset of $Range(L)$.

(c) If $S = \{v_1, v_2, \ldots, v_m\}$ is **dependent** in V, and linear map $L : V \to W$ is **injective**, you can be sure that $L(S) = \{L(v_1), \ldots, L(v_m)\}$ is **dependent**.

(d) In W let $T = \{w_1, w_2, \ldots, w_{m-1}\}$ and $T' = \{w_1, w_2, \ldots, w_{m-1}, w_m\}$ where the last vector $w_m \in \langle T \rangle$ is a linear combination of the previous vectors. The relationship is $\langle T \rangle = \langle T' \rangle$ since w_m is redundant in T'.

(e) If $S = \{v_1, v_2, \ldots, v_n\}$ is **independent** in V and $v \in \langle S \rangle$, then $T = \{v_1, v_2, \ldots, v_n, v\}$ is **dependent** since v is redundant, so $\langle T \rangle = \langle S \rangle$ has basis S so $dim(\langle T \rangle) = n$.

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(3) (10 Points, 2 points each)

(a) Use $\dim(V) = \dim(Ker(L)) + \dim(Range(L))$. L injective means $\dim(Ker(L)) = 0$ so $\dim(V) = \dim(\mathbb{F}_3^3) = 9 = \dim(Range(L)) = dim(\mathbb{F}^9)$, so $Range(L) = \mathbb{F}^9$ so L is surjective. L is thus bijective, invertible and an isomorphism.

(b) $L : \mathbb{F}^3 \to \mathbb{F}^8$ so $3 = \dim(Ker(L)) + \dim(Range(L))$ and $0 \le dim(Ker(L)) \le 3$ so $0 \le dim(Range(L)) \le 3$.

(c) $L : \mathbb{F}^7 \to \mathbb{F}^4$ so $7 = \dim(\mathbb{F}^7) = \dim(Ker(L)) + \dim(Range(L))$ and $0 \le \dim(Range(L)) \le 4$ so $3 \le dim(Ker(L)) \le 7$.

(d) If $A \in \mathbb{F}_n^n$ and the homogeneous linear system $AX = 0$ has **non-trivial** solutions, then you can be sure that $rank(A) < n$ so $\det(A) = 0$.

(e) Let $A \in \mathbb{F}_n^n$ be **invertible** and suppose a **non-zero** $X \in \mathbb{F}^n$ satisfies $AX = \lambda X$ for some $\lambda \in \mathbb{F}$. Then $\lambda \neq 0$ because otherwise $AX = \mathbf{0}$ has a **non-trivial** solution so $Rank(A) < n$ so A could not be invertible.

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(4) (10 Points) (a) (5 Pts) To solve $x_1 v_1 + x_2 v_2 + x_3 v_3 = v$ row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 2 & 1 & b \\ 1 & 3 & 1 & c \\ & S & & v \end{array}\right] \quad \text{to} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & a+b-c \\ 0 & 1 & 0 & -b+c \\ 0 & 0 & 1 & -a+2b-c \\ & I_3 & & X \end{array}\right] \quad \text{so } [v]_S = \begin{bmatrix} a+b-c \\ -b+c \\ -a+2b-c \end{bmatrix}.$$

(b) (5 Pts) To write p as a linear combination from T, solve the equation $x_1(t^2) + x_2(t^2 + t) + x_3(t^2 + t + 1) = at^2 + bt + b$.
Compare coefficients of each power of t on both sides to get the linear system $x_1 + x_2 + x_3 = a$, $x_2 + x_3 = b$, $x_3 = c$. To solve that, row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \\ & T & & p \end{array}\right] \quad \text{to} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \\ & I_3 & & X \end{array}\right].$$ So $[p]_T = \begin{bmatrix} a-b \\ b-c \\ c \end{bmatrix}$. We can check that $(a-b)t^2 + (b-c)(t^2+t) + c(t^2+t+1) = at^2 + bt + c$.

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(5) (10 Points, 2 points each) Answer each question separately.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ is invertible and $A^{-1} = \begin{bmatrix} -2 & 3 & 4 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}$ because

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ & A & & & I_3 & \end{array}\right] \quad \text{reduces to} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 \\ & I_3 & & & A^{-1} & \end{array}\right].$$

(b) If $E \in \mathbb{R}^n_n$ is an elementary matrix corresponding to an elementary **switcher** row operation then $\det(E) = -1$.

(c) If $A \in \mathbb{F}^n_n$ has characteristic polynomial $\det(\lambda I_n - A) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r}$ with r distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ then $k_1 + k_2 + \cdots + k_r = n$ is the degree of that polynomial.

(d) With notation as in part (c), let $A_{\lambda_i} = \{X \in \mathbb{F}^n \mid AX = \lambda_i X\}$ be the λ_i eigenspace of A. The **geometric multiplicity** of A is $g_i = dim(A_{\lambda_i})$.

(e) If $A^T = A^{-1}$ for $A \in \mathbb{F}^n_n$, then $\det(A) = \det(A^T) = \det(A^{-1}) = \frac{1}{\det(A)}$ so $(\det(A))^2 = 1$ so $\det(A) = \pm 1$.

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(6) (10 Points)

(a) (2 Pts) $L(S)$: $L(v_1) = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, $L(v_2) = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$. Then $[T \mid L(S)] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 4 & -5 \\ & T & & \multicolumn{2}{c}{L(S)} \end{array}\right]$

so ${}_T[L]_S = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 4 & -5 \end{bmatrix}$.

(b) (3 Pts) Find $L(S')$: $L(v_1') = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}$, $L(v_2') = \begin{bmatrix} 11 \\ 2 \\ -11 \end{bmatrix}$. Row reduce

$$\left[\begin{array}{ccc|cc} 1 & 0 & 1 & 8 & 11 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -6 & -11 \\ & T' & & \multicolumn{2}{c}{L(S')} \end{array}\right] \text{ to } \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 7 & 13 \\ 0 & 1 & 0 & -7 & -9 \\ 0 & 0 & 1 & 1 & -2 \\ & I_3 & & \multicolumn{2}{c}{{}_{T'}[L]_{S'}} \end{array}\right] \text{ so } {}_{T'}[L]_{S'} = \begin{bmatrix} 7 & 13 \\ -7 & -9 \\ 1 & -2 \end{bmatrix}.$$

(c) (3 Pts) ${}_SP_{S'} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ since S and T are standard bases.

To get ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$, row reduce

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ & T' & & & T & \end{array}\right] \text{ to } \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ & I_3 & & & {}_{T'}Q_T & \end{array}\right] \text{ so } {}_{T'}Q_T = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

(d) (2 Pts) $({}_{T'}Q_T)({}_T[L]_S)({}_SP_{S'}) =$

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 6 \\ -3 & -2 \\ 7 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 13 \\ -7 & -9 \\ 1 & -2 \end{bmatrix} =$$

${}_{T'}[L]_{S'}$ is the relationship, as it should be according to a theorem we proved in class.

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(7) (15 Points) Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(a) (5 points) $Char_A(\lambda) = \det(\lambda I_4 - A) = \det(A - \lambda I_4) =$

$$\det \begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} = \det \begin{bmatrix} -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & \lambda \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} =$$

$$\lambda^2 \det \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} = \lambda^2 \det \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = \lambda^2 (\lambda-2)^2$$

(b) (2 points) The **eigenvalues** of A are $\lambda_1 = 0$ and $\lambda_2 = 2$ with corresponding **algebraic multiplicities** $k_1 = 2$ and $k_2 = 2$.

(c) (6 points) For each eigenvalue λ_i of A, find a **basis** of its eigenspace A_{λ_i} by solving the homogeneous linear system $[A - \lambda_i I_4 | 0_1^4]$. For $\lambda_1 = 0$, row reduce $[A|0_1^4] =$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -r \\ x_2 = -s \\ x_3 = r \in \mathbb{F} \\ x_4 = s \in \mathbb{F} \end{array}, \text{ then}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} -r \\ -s \\ r \\ s \end{bmatrix} \in \mathbb{F}^4 \;\middle|\; r, s \in \mathbb{F} \right\} \text{ has basis } T_1 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda_2 = 2$, row reduce $[A - 2I_4 | 0_1^4] =$

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r \\ x_2 = s \\ x_3 = r \in \mathbb{F} \\ x_4 = s \in \mathbb{F} \end{array}, \text{ then}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} r \\ s \\ r \\ s \end{bmatrix} \in \mathbb{F}^4 \;\middle|\; r, s \in \mathbb{F} \right\} \text{ has basis } T_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(d) (2 points) A is diagonalizable because $T = T_1 \cup T_2$ is an eigenbasis of F^4. The geometric multiplicities are $g_1 = 2 = k_1$ and $g_2 = 2 = k_2$. A is similar to the diagonal matrix $D = Diag(\lambda_1, \lambda_1, \lambda_2, \lambda_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.
