Math 304-6 Linear Algebra Spring 2025 Exam 3 Feingold

SHOW ALL NECESSARY WORK. Note: A^T means the transpose of A.

(1) (15 Points, 3 points each) Answer each question separately with a **brief justification**.

(a) If $X, Y \in \mathbb{R}^n$ and $\theta_{X,Y}$ is the angle between X and Y, what is the formula for $\cos(\theta_{X,Y})$ in terms of the standard dot product on $\mathbb{R}^n$?

(b) Let $L : V \to V$ be **invertible** and suppose $v \in V$ is an eigenvector for L with eigenvalue $0 \neq \lambda \in \mathbb{F}$. Show that v is an eigenvector for L^{-1} with eigenvalue λ^{-1}. Do not assume L is diagonalizable.

(c) Suppose $A \in \mathbb{F}^n_n$ and $Char_A(\lambda) = \det(\lambda I_n - A) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r}$ with r distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ in $\mathbb{F}$. Let T_i be a basis of eigenspace A_{λ_i}. What is the **most** you can say in general about the union $T = T_1 \cup \cdots T_r$?

(d) With notation as in part (c), what property of set T means that A is diagonalizable?

(e) Let $E \in \mathbb{F}^n_n$ be an elementary matrix corresponding to an elementary **adder** row operation. What can you say about $\det(E)$?

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(2) (15 Points, 3 points each) Answer each question separately. **Show all work**.

(a) Find $\det \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix}$.

(b) If $A \in \mathbb{F}^n_n$ has $\det(A) = 5$ what is $\det(3A^{-1})$?

(c) If $A^T = -A$ for $A \in \mathbb{F}^n_n$ where n is **odd**, what is the **most** you can say about $det(A)$?

(d) Suppose $A, B \in \mathbb{F}^n_n$ are **similar**, that is, $B = P^{-1}AP$ for some invertible $P \in \mathbb{F}^n_n$. What is the relationship between $\det(A)$ and $\det(B)$?

(e) Suppose $A = A^T \in \mathbb{R}^n_n$ and $\lambda \neq \mu$ in $\mathbb{R}$ are eigenvalues of A with corresponding eigenvectors, X and Y, so $AX = \lambda X$ and $AY = \mu Y$. What is the **most** you can say about $X \cdot Y$?

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(3) (15 Points) Let $A = \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \in \mathbb{R}^4_4$.

(a) (4 Pts) Find the **characteristic polynomial** of A, $\det(A - tI_4)$, find all **eigenvalues**, $\lambda_i \in \mathbb{R}$, of A and the corresponding **algebraic multiplicities**, k_i. **Hint**: Using the last row of $A - tI_4$, adder row operations simplify the first three rows so that linear factors come out of $det(A - tI_4)$.

(b) (6 Pts) For each eigenvalue, λ_i, of A, find a **basis** for the **eigenspace**, A_{λ_i}, and the **geometric multiplicity** $g_i = \dim(A_{\lambda_i})$.

(c) (5 Pts) Determine whether or not A is **diagonalizable** over $\mathbb{R}$. If it is, find real matrices D and P such that $D = P^{-1}AP$ is diagonal. If not, explain why.

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(4) (20 Points) Let $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$ and $T = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be an ordered basis for a subspace W of $\mathbb{R}^4$.

(a) Use the Gram-Schmidt orthogonalization process to convert T into an **orthogonal** basis of W, $T' = \{w_1', w_2', w_3'\}$, where

$$w_1' = w_1, \qquad w_2' = w_2 - Proj_{\langle w_1' \rangle}(w_2) \quad \text{and} \quad w_3' = w_3 - Proj_{\langle w_1', w_2' \rangle}(w_3).$$

After using these formulas, **please rescale** w_i' if necessary to **avoid fractions**. Be sure to check your work by verifying that $w_i' \cdot w_j' = 0$ for all $1 \le i < j \le 3$.

(b) Use your answer to part (a) to **find the projection**, $Proj_W(v) = \sum_{i=1}^{3} x_i w_i'$ of a general vector v into the subspace W. The **first step** is to find the coefficients, x_1, x_2, x_3, which are uniquely determined by the condition that $v - Proj_W(v)$ is orthogonal to W, that is, $(v - Proj_W(v)) \cdot w_j' = 0$ for $1 \le j \le 3$. After you have found the coefficients, **write the formula** for $Proj_W(v)$ in terms of a, b, c, d and **simplify all expressions** to get your final answer. **Hint**: To check your simplified formula, see if it satisfies $Proj_W(w_j) = w_j$ for $1 \le j \le 3$.

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(5) (10 Points) Let $W = \langle T \rangle$ from problem (4).

(a) (5 Pts) Find a **basis** for $W^\perp = \{X \in \mathbb{R}^4 \mid w_i \cdot X = 0,\ 1 \le i \le 3\}$.
(b) (5 Pts) Show that $W + W^\perp = \mathbb{R}^4$ is an **orthogonal direct sum**.

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(1) (15 Points, 3 points each) Answer each question separately with a **brief justification**.

(a) If $X, Y \in \mathbb{R}^n$ and $\theta_{X,Y}$ is the angle between X and Y, the formula is $\cos(\theta_{X,Y}) = \dfrac{X \cdot Y}{\sqrt{X \cdot X}\sqrt{Y \cdot Y}}$ since $||X|| = \sqrt{X \cdot X}$.

(b) Let $L : V \to V$ be **invertible** and suppose $v \in V$ is an eigenvector for L with eigenvalue $0 \neq \lambda \in \mathbb{F}$. Show that v is an eigenvector for L^{-1} with eigenvalue λ^{-1}. Do not assume L is diagonalizable. We know $v = L^{-1}(L(v)) = L^{-1}(\lambda v) = \lambda L^{-1}(v)$ so $\lambda^{-1}v = L^{-1}(v)$ shows that v is an eigenvector for L^{-1} with eigenvalue λ^{-1}.

(c) Suppose $A \in \mathbb{F}^n_n$ and $Char_A(\lambda) = \det(\lambda I_n - A) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r}$ with r distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ in $\mathbb{F}$. Let T_i be a basis of eigenspace A_{λ_i}. The **most** you can say in general about the union $T = T_1 \cup \cdots\ T_r$ is that it is an **independent** set of $g_1 + \cdots + g_r$ eigenvectors.

(d) With notation as in part (c), the property that T is a basis of $\mathbb{F}^n$ means that A is diagonalizable. Other answers are: T contains n vectors, or T spans $\mathbb{F}^n$.

(e) Let $E \in \mathbb{F}^n_n$ be an elementary matrix corresponding to an elementary **adder** row operation. Then $\det(E) = 1$.

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(2) (15 Points, 3 points each) Answer each question separately. **Show all work**.

(a) $$\det \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} = \det \begin{bmatrix} -2 & 0 & 0 & 8 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 2 \\ 5 & -5 & -5 & -7 \end{bmatrix} =$$
$$(-2)^3 \det \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 5 & -5 & -5 & -7 \end{bmatrix} = (-2)^3 \det \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = -24.$$

(b) If $A \in \mathbb{F}^n_n$ has $\det(A) = 5$ then $\det(3A^{-1}) = 3^n \det(A^{-1}) = \dfrac{3^n}{5}$.

(c) If $A^T = -A$ for $A \in \mathbb{F}^n_n$ where n is **odd**, then $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$ since n is odd, so $\det(A) = 0$.

(d) Suppose $A, B \in \mathbb{F}^n_n$ are **similar**, that is, $B = P^{-1}AP$ for some invertible $P \in \mathbb{F}^n_n$. Then $\det(A) = \det(B)$ since $\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(P)^{-1}\det(A)\det(P) = \det(A)$.

(e) Suppose $A = A^T \in \mathbb{R}^n_n$ and $\lambda \neq \mu$ in $\mathbb{R}$ are eigenvalues of A with corresponding eigenvectors, X and Y, so $AX = \lambda X$ and $AY = \mu Y$. We can say $X \cdot Y = 0$ was proved in class as follows. $\lambda(X \cdot Y) = (\lambda X) \cdot Y = (AX) \cdot Y = X \cdot (A^T Y) = X \cdot (AY) = X \cdot (\mu Y) = \mu(X \cdot Y)$ so $(\lambda - \mu)(X \cdot Y) = 0$ and since $\lambda \neq \mu$, we get $X \cdot Y = 0$.

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(3) (15 Points) Let $A = \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \in \mathbb{R}^4_4$.

(a) (4 Points) The characteristic polynomial is $Char_A(t) = \det(tI_4 - A) = \det(A - tI_4)$

$$\det \begin{bmatrix} 18-t & -20 & -20 & -20 \\ 5 & -7-t & -5 & -5 \\ 5 & -5 & -7-t & -5 \\ 5 & -5 & -5 & -7-t \end{bmatrix} = \det \begin{bmatrix} -t-2 & 0 & 0 & 4t+8 \\ 0 & -t-2 & 0 & t+2 \\ 0 & 0 & -t-2 & t+2 \\ 5 & -5 & -5 & -7-t \end{bmatrix}$$

$$= (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 5 & -5 & -5 & -7-t \end{bmatrix} = (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -5 & -5 & -t+13 \end{bmatrix}$$

$$= (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -5 & -t+8 \end{bmatrix} = (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -t+3 \end{bmatrix}$$

$= (t+2)^3\ (t-3)$. (Constant term, $Char_A(0) = -24 = \det(A)$ answers problem 2(a).

So the eigenvalues are $\lambda_1 = -2$ with algebraic multiplicity $k_1 = 3$ and $\lambda_2 = 3$ with algebraic multiplicity $k_2 = 1$.

(b) (6 Points) Check the $\lambda_1 = -2$ eigenspace first since the algebraic multiplicity $k_1 = 3$. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $\lambda = -2$ to $A - \lambda I_4$. Row reduce

$$\left[\begin{array}{cccc|c} 20 & -20 & -20 & -20 & 0 \\ 5 & -5 & -5 & -5 & 0 \\ 5 & -5 & -5 & -5 & 0 \\ 5 & -5 & -5 & -5 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r+s+t \\ x_2 = r \in \mathbb{R} \\ x_3 = s \in \mathbb{R} \\ x_4 = t \in \mathbb{R} \end{array}, \text{ then}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} r+s+t \\ r \\ s \\ t \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; r,s,t \in \mathbb{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ so } g_1 = 3.$$

Since there will be one more independent eigenvector from the other eigenvalue, we will have the necessary four eigenvectors to form a basis for $\mathbb{R}^4$, so this A is diagonalizable.

Now find the $\lambda_2 = 3$ eigenspace. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $\lambda = 3$ to $A - \lambda I_4$. Row reduce

$$\left[\begin{array}{cccc|c} 15 & -20 & -20 & -20 & 0 \\ 5 & -10 & -5 & -5 & 0 \\ 5 & -5 & -10 & -5 & 0 \\ 5 & -5 & -5 & -10 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 4r \\ x_2 = r \\ x_3 = r \\ x_4 = r \in \mathbb{R} \end{array}, \text{ then}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} 4r \\ r \\ r \\ r \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; r \in \mathbb{R} \right\} \quad \text{has basis} \quad \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ so } g_2 = 1.$$

(c) (5 Points) Therefore, $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, ${}_TD_T = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

and

$P = {}_SP_T = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = {}_TP_S = \begin{bmatrix} -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix}$. Check:

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -4 & -2 & -2 \\ 2 & -2 & -4 & -2 \\ 2 & -2 & -2 & -4 \\ 3 & -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = {}_TD_T. \end{aligned}$$

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(4) (20 Points) The set $T = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an ordered basis for a subspace W of $\mathbb{R}^4$.

(a) Use the Gram-Schmidt orthogonalization process to convert T into an **orthogonal** basis T' for W.

Solution: (10 Pts) First let $w_1' = w_1$, then

$$w_2' = w_2 - \frac{w_2 \cdot w_1'}{w_1' \cdot w_1'} w_1' = w_2 - \frac{12}{4}\, w_1' = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \text{ and check that } w_1' \cdot w_2' = 0.$$

Let

$$w_3' = w_3 - \frac{w_3 \cdot w_1'}{w_1' \cdot w_1'} w_1' - \frac{w_3 \cdot w_2'}{w_2' \cdot w_2'} w_2' = w_3 - \frac{3}{4}\, w_1' - \frac{1}{4}\, w_2' = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

We may double this vector and it will still be orthogonal to the previous two vectors, so

$$T' = \left\{ w_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2' = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, w_3' = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is an orthogonal basis for W obtained by the Gram-Schmidt orthogonalization process.

(b) Use your answer to part (a) to **find the projection**, $Proj_W(v) = \sum_{i=1}^{3} x_i w_i'$ of a general vector v into the subspace W. The coefficients, x_1, x_2, x_3, are uniquely determined by the condition that $v - Proj_W(v)$ is orthogonal to W, that is, $(v - Proj_W(v)) \cdot w_j' = 0$ for $1 \leq j \leq 3$. After you have found the coefficients, write the formula for $Proj_W(v)$ in terms of a, b, c, d.

Solution: (10 Pts) The conditions mean that $v \cdot w_j' = Proj_W(v) \cdot w_j' = x_j(w_j' \cdot w_j')$ for $1 \leq j \leq 3$ since T' is an orthogonal set. This says $x_j = \dfrac{v \cdot w_j'}{w_j' \cdot w_j'}$ so from part (a),

$$x_1 = \frac{a+b+c+d}{4}, \quad x_2 = \frac{-a+b-c+d}{4}, \quad x_3 = \frac{-a+c}{2} \quad \text{so}$$

$$Proj_W(v) = \frac{(a+b+c+d)}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{(-a+b-c+d)}{4} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{(-a+c)}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} (a+b+c+d) - (-a+b-c+d) - (-2a+2c) \\ (a+b+c+d) + (-a+b-c+d) \\ (a+b+c+d) - (-a+b-c+d) + (-2a+2c) \\ (a+b+c+d) + (-a+b-c+d) \end{bmatrix} = \begin{bmatrix} a \\ (b+d)/2 \\ c \\ (b+d)/2 \end{bmatrix}.$$

This formula does satisfy $Proj_W(w_j) = w_j$ for $1 \leq j \leq 3$.

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(5) (10 Points) Let $W = \langle T \rangle$ from problem (4).

(a) (5 Pts) Find a **basis** for $W^\perp = \{X \in \mathbb{R}^4 \mid w_i \cdot X = 0,\ 1 \leq i \leq 3\}$.

Solution: (5 Pts) Find $W^\perp$ by solving $[A|0_1^3]$ where $Row_i(A) = w_i^T$. Row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 4 & 2 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 0 \\ x_2 = -r \\ x_3 = 0 \\ x_4 = r \in \mathbb{R} \end{array} \text{ so } W^\perp \text{ has basis } \left\{ w_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) (5 Pts) Show that $W + W^\perp = \mathbb{R}^4$ is an **orthogonal direct sum**.

Solution: (5 Pts) $T \cup \{w_4\}$ is a basis of $\mathbb{R}^4$ since T is independent and $w_4 \notin \langle T \rangle$. Also, $w_4 \perp T$, so the sum is orthogonal, and therefore, direct. In fact, $W \cap W^\perp = \{0_1^4\}$ because only the zero vector is orthogonal to itself.
