NAME (Printed): ____

Math 304-6 Linear Algebra Spring 2025 Quiz 10 Feingold

No reasons needed to justify your answers.

$\mathbb{R}$ is the real numbers, $\mathbb{C}$ is the complex numbers.
For any $A = [a_{ij}] \in \mathbb{C}^m_n$, the complex conjugate of A is $\overline{A} = [\bar{a}_{ij}]$.

---

(1) (2 Pts) If $A, B \in \mathbb{R}^n_n$ and $(AX)\cdot Y = X\cdot(BY)$ for all $X, Y \in \mathbb{R}^n$, then the **relationship** between A and B is:

---

(2) (2 Pts) For $v_1, \cdots, v_k \in \mathbb{R}^n$, the **most general** situation when $||v_1 + \cdots + v_k||^2 = ||v_1||^2 + \cdots + ||v_k||^2$ is **guaranteed** is when the relationship among these vectors is:

---

(3) (2 Pts) The Triangle Inequality in $\mathbb{R}^n$, $||X + Y|| \leq ||X|| + ||Y||$ for any $X, Y \in \mathbb{R}^n$, implies that for any $v_1, \cdots, v_k \in \mathbb{R}^n$ we have $||v_1 + \cdots + v_k|| \leq$

---

(4) (2 Pts) For $Z, W \in \mathbb{C}^n$ we have the dot product $Z \cdot W = Z^T\overline{W}$, where $\overline{W}$ is the complex conjugate of W. If $A, B \in \mathbb{C}^n_n$ and $(AZ) \cdot W = Z \cdot (BW)$ for all $Z, W \in \mathbb{C}^n$, then the **relationship** between A and B is:

---

(5) (2 Pts) A matrix $A \in \mathbb{C}^n_n$ is called **unitary** when $\overline{A}^T = A^{-1}$. Using the fact that $\det(\overline{A}) = \overline{\det(A)}$ for any matrix A, we can say that for A unitary, $\det(A) = z = a + b\mathbf{i} \in \mathbb{C}$ must satisfy the condition on a and b that:

Math 304-6 Linear Algebra Spring 2025 Quiz 10 Solutions Feingold

No reasons were needed to justify your answers but justifications were included in the solutions for your understanding.

$\mathbb{R}$ is the real numbers, $\mathbb{C}$ is the complex numbers.
For any $A = [a_{ij}] \in \mathbb{C}^m_n$, the complex conjugate of A is $\overline{A} = [\bar{a}_{ij}]$.

---

(1) (2 Pts) If $A, B \in \mathbb{R}^n_n$ and $(AX)\cdot Y = X\cdot(BY)$ for all $X, Y \in \mathbb{R}^n$, then the **relationship** between A and B is: $B = A^T$.

Justification: $(AX)\cdot Y = (AX)^T Y = X^T A^T Y = X\cdot(A^T Y) = X\cdot(BY)$ so $X\cdot(A^T Y - BY) = 0$ is true for all $X, Y \in \mathbb{R}^n$. This being true for all $X \in \mathbb{R}^n$ gives $0^n_1 = A^T Y - BY = (A^T - B)Y$, and that being true for all $Y \in \mathbb{R}^n$ gives $A^T - B = 0^n_n$ so $B = A^T$.

---

(2) (2 Pts) For $v_1, \cdots, v_k \in \mathbb{R}^n$, the **most general** situation when $||v_1 + \cdots + v_k||^2 = ||v_1||^2 + \cdots + ||v_k||^2$ is **guaranteed** is when the relationship among these vectors is: $v_i \cdot v_j = 0$ for $1 \leq i < j \leq k$, that is, when $\{v_1, \cdots, v_k\}$ is an **orthogonal** set.

Justification: This is the generalized Pythagorean Theorem.

---

(3) (2 Pts) The Triangle Inequality in $\mathbb{R}^n$, $||X + Y|| \leq ||X|| + ||Y||$ for any $X, Y \in \mathbb{R}^n$, implies that for any $v_1, \cdots, v_k \in \mathbb{R}^n$ we have $||v_1 + \cdots + v_k|| \leq ||v_1|| + \cdots + ||v_k||$.

Justification: Follows from the Triangle Inequality by induction on k.

---

(4) (2 Pts) For $Z, W \in \mathbb{C}^n$ we have the dot product $Z \cdot W = Z^T\overline{W}$, where $\overline{W}$ is the complex conjugate of W. If $A, B \in \mathbb{C}^n_n$ and $(AZ)\cdot W = Z\cdot(BW)$ for all $Z, W \in \mathbb{C}^n$, then the **relationship** between A and B is: $B = \overline{A}^T$.

Justification: $(AZ)\cdot W = (AZ)^T\overline{W} = Z^T A^T \overline{W} = Z^T\overline{(\overline{A}^T W)} = Z\cdot(\overline{A}^T W) = Z\cdot(BW)$ true for all $Z, W \in \mathbb{C}^n$. The rest of the argument is as in problem (1).

---

(5) (2 Pts) A matrix $A \in \mathbb{C}^n_n$ is called **unitary** when $\overline{A}^T = A^{-1}$. Using the fact that $\det(\overline{A}) = \overline{\det(A)}$ for any matrix A, we can say that for A unitary, $\det(A) = z = a + b\mathbf{i} \in \mathbb{C}$ must satisfy the condition $z\bar{z} = a^2 + b^2 = 1$.

Justification: $\overline{A}^T = A^{-1}$ means $I_n = A\overline{A}^T$ so

$$1 = \det(I_n) = \det(A\overline{A}^T) = \det(A)\det(\overline{A}^T) = \det(A)\overline{\det(A)}$$

$$= z\bar{z} = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2.$$

Then the condition on z is that $a^2 + b^2 = 1$, which is a circle in $\mathbb{C}$, which could be written as $\{z = a + b\mathbf{i} = \cos(\phi) + \mathbf{i}\sin(\phi) \in \mathbb{C} \mid 0 \leq \phi \leq 2\pi\}$.
