NAME (Printed): ____

Math 304-6 Linear Algebra Spring 2025 Quiz 1 Feingold

INSTRUCTIONS: Show all necessary work for each problem.

Let $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 2 & -4 & 7 \\ 2 & 3 & 1 & 0 \\ 5 & 4 & 6 & -7 \end{bmatrix}$ be the coefficient matrix of the linear system $AX = 0^4_1$.

(1) (5 Points) Row reduce $[A|0^4_1]$ to **Reduced Row Echelon Form** (RREF) to find the solution set $\{X \in \mathbb{F}^4 \mid AX = 0^4_1\}$ in terms of some **free variables** in $\mathbb{F}$.

(2) (5 Points) Let $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in \mathbb{F}^4$. Row reduce $[A|B]$ to Reduced Row Echelon Form (RREF) to find the **consistency conditions** on the entries of B required for the linear system $AX = B$ to be **consistent**.

Math 304-6 Linear Algebra Spring 2025 Quiz 1 Solutions Feingold

1. (5 Points) The row reduction of $[A|0_1^4]$ to RREF needed to solve $AX = 0_1^4$ is:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ -1 & 2 & -4 & 7 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 5 & 4 & 6 & -7 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 3 & -3 & 6 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = -2r + 3s \\ x_2 = r - 2s \\ x_3 = r \in \mathbb{F} \\ x_4 = s \in \mathbb{F}. \end{array}$$

The solution set in terms of **free variables** is $\left\{ X = \begin{bmatrix} -2r + 3s \\ r - 2s \\ r \\ s \end{bmatrix} \in \mathbb{F}^4 \;\middle|\; r, s \in \mathbb{F} \right\}$.

In the first step of the row reduction the row operations used were: $R_1 + R_2 \to R_2$, $-2R_1 + R_3 \to R_4$ and $-5R_1 + R_4 \to R_4$. In the second step the row operations used were: $-R_3 + R_1 \to R_1$, $-3R_3 + R_2 \to R_2$, $R_3 + R_4 \to R_4$, $R_2 \leftrightarrow R_3$.

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2. (5 Points) For $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in \mathbb{F}^4$ **consistency conditions** on the entries of B for $AX = B$ are found by row reduction of $[A|B]$. The consistency conditions come from rows with all zeros on the left side of the RREF:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & b_1 \\ -1 & 2 & -4 & 7 & b_2 \\ 2 & 3 & 1 & 0 & b_3 \\ 5 & 4 & 6 & -7 & b_4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & b_1 \\ 0 & 3 & -3 & 6 & b_2 + b_1 \\ 0 & 1 & -1 & 2 & b_3 - 2b_1 \\ 0 & -1 & 1 & 2 & b_4 - 5b_1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 3b_1 - b_3 \\ 0 & 1 & -1 & 2 & -2b_1 + b_3 \\ 0 & 0 & 0 & 0 & 7b_1 + b_2 - 3b_3 \\ 0 & 0 & 0 & 0 & -7b_1 + b_3 + b_4 \end{array}\right] \begin{array}{c} \text{is consistent iff} \\ 0 = 7b_1 + b_2 - 3b_3 \\ \text{and} \\ 0 = -7b_1 + b_3 + b_4. \end{array}$$

Note that these two conditions are equivalent to the two conditions $b_3 + b_4 = 7b_1$ and $b_2 + b_4 = 2b_3$, and these conditions are satisfied by each of the four columns of A.
Can you explain why $AX = Col_j(A)$ is consistent for $1 \leq j \leq 4$?
