NAME (Printed): ______________________________

Math 304-6 Linear Algebra Spring 2025 Quiz 5 Feingold

INSTRUCTIONS: Show all calculations needed to justify your answers.

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$ and let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where $v_j = Col_j(A) \in \mathbb{R}^4$.

Note that $AX = \theta = 0_1^4$ is solved by row reducing

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & 4 & 5 & 6 & 7 & 0 \\ 4 & 5 & 6 & 7 & 8 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = r + 2s + 3t \\ x_2 = -2r - 3s - 4t \\ x_3 = r \in \mathbb{R} \\ x_4 = s \in \mathbb{R} \\ x_5 = t \in \mathbb{R} \end{array} .$$

(1) (4 Pts) Use that information to write a **single equation** that gives all **dependence relations** among the vectors in S in terms of the three free variables.

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(2) (3 Pts) For **each free variable** found in the answer to part (1), get an equation that expresses a vector in S as a linear combination of **previous** vectors in S.

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(3) (3 Pts) Use your answer to part (2) to remove **redundant** vectors from S and give the smallest subset $T \subset S$ such that T is **independent** and $\langle T \rangle = \langle S \rangle$, that is, the span of T is the same as the span of S.

Math 304-6 Linear Algebra Spring 2025 Quiz 5 Solutions Feingold

INSTRUCTIONS: Show all calculations needed to justify your answers.

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$ and let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where $v_j = Col_j(A) \in \mathbb{R}^4$.

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$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & 4 & 5 & 6 & 7 & 0 \\ 4 & 5 & 6 & 7 & 8 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r + 2s + 3t \\ x_2 = -2r - 3s - 4t \\ x_3 = r \in \mathbb{R} \\ x_4 = s \in \mathbb{R} \\ x_5 = t \in \mathbb{R} \end{array}.$$

(1) (4 Pts) Use that information to write a **single equation** that gives all **dependence relations** among the vectors in S in terms of the three free variables.

Solution: To find all dependence relations on S we must solve the homogeneous linear system $\sum_{j=1}^{5} x_j v_j = \theta$, which is $AX = 0_1^4$. The solutions found by row reduction of $[A|0_1^4]$ tell us that all dependence relations on S are:

$$(r + 2s + 3t)v_1 + (-2r - 3s - 4t)v_2 + rv_3 + sv_4 + tv_5 = \theta \quad \text{for any } r, s, t \in \mathbb{R}.$$

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(2) (3 Pts) For **each free variable** found in the answer to part (1), get an equation that expresses a vector in S as a linear combination of **previous** vectors in S.

Solution:

For $r = 1,\ s = 0,\ t = 0$ we get $1v_1 - 2v_2 + 1v_3 = \theta$ so $v_3 = -v_1 + 2v_2$.

For $r = 0,\ s = 1,\ t = 0$ we get $2v_1 - 3v_2 + 1v_4 = \theta$ so $v_4 = -2v_1 + 3v_2$.

For $r = 0,\ s = 0,\ t = 1$ we get $3v_1 - 4v_2 + 1v_5 = \theta$ so $v_5 = -3v_1 + 4v_2$.

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(3) (3 Pts) Use your answer to part (2) to remove **redundant** vectors from S and give the smallest subset $T \subset S$ such that T is **independent** and $\langle T \rangle = \langle S \rangle$, that is, the span of T is the same as the span of S.

Solution: The answers to part (2) show that $v_3, v_4, v_5 \in \langle v_1, v_2 \rangle$ so the last three vectors are redundant in S and $T = \{v_1, v_2\}$ has the same span as S. T is independent since the row reduction in part (1) done only with the first two columns of A has only the trivial solution.
