NAME (Printed): ______________________________

Math 304-6 Linear Algebra Spring 2025 Quiz 6 Feingold

Show all calculations and reasons needed to justify your answers.

(1) (4 Pts) Let $V = \mathbb{F}_2^2$. Find a **basis** for subspace $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}_2^2 \;\middle|\; a - b + 2c - 3d = 0 \right\}$.

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(2) (2 Pts) Suppose $S = \{u_1, u_2, \cdots, u_m\}$ is an **independent** subset of a vector space U and $u \in \langle S \rangle$. Then $T = \{u_1, u_2, \cdots, u_m, u\}$ **must** be (Circle one answer):
independent **dependent** since (justify your answer)

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(3) (2 Pts) Suppose $S = \{v_1, v_2, \cdots, v_k\}$ **spans** V. What is the relationship between k and $dim(V)$? Why?

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(4) (2 Pts) Suppose $S = \{u_1, u_2, \cdots, u_m\}$ is an **independent** subset of U. What is the relationship between m and $dim(U)$? Why?

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INSTRUCTIONS: Show all calculations and reasons needed to justify your answers.

(1) (4 Pts) Let $V = \mathbb{F}_2^2$. Find a **basis** for subspace $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}_2^2 \,\middle|\, a - b + 2c - 3d = 0 \right\}$.

Solution: Since $a = b - 2c + 3d$ we can write any vector in W as

$$\begin{bmatrix} b-2c+3d & b \\ c & d \end{bmatrix} = b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = bw_1 + cw_2 + dw_3$$

so $W = \langle w_1, w_2, w_3 \rangle$. Also, $\{w_1, w_2, w_3\}$ is independent since $bw_1 + cw_2 + dw_3 = 0_2^2$ means $\begin{bmatrix} b-2c+3d & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $b = c = d = 0$. Then $\{w_1, w_2, w_3\}$ is a basis of W.

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(2) (2 Pts) Suppose $S = \{u_1, u_2, \cdots, u_m\}$ is an **independent** subset of a vector space U and $u \in \langle S \rangle$. Then $T = \{u_1, u_2, \cdots, u_m, u\}$ **must** be (Circle one answer):

independent **dependent** since (justify your answer)

Solution: T must be **dependent** since $u \in \langle S \rangle$ means it is **redundant** in T. In fact, $u \in \langle S \rangle$ means $u = \sum_{i=1}^m a_i u_i$ for some $a_i \in \mathbb{F}$, we have the dependence relation $1u - \sum_{i=1}^m a_i u_i = \theta$ among the vectors in T.

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(3) (2 Pts) Suppose $S = \{v_1, v_2, \cdots, v_k\}$ **spans** V. What is the relationship between k and $dim(V)$? Why?

Solution: $dim(V) \leq k$ because S may be either independent or dependent. If S is independent, then S is a basis for V so $dim(V) = k$. But if S is dependent, there are redundant vectors which can be removed, leaving an independent subset of less than k vectors that span V, a basis for V. So in general S can be cut down to a basis of V and the most we can say is that $dim(V) \leq k$.

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(4) (2 Pts) Suppose $S = \{u_1, u_2, \cdots, u_m\}$ is an **independent** subset of U. What is the relationship between m and $dim(U)$? Why?

Solution: $dim(U) \geq m$ because S may span U or it may not. If S spans U then it is a basis for U so $dim(U) = m$. If S does not span U then it can be extended to a basis of U, so in general, the most we can say is that $dim(U) \geq m$.
