NAME (Printed): ____

Math 304-6 Linear Algebra Spring 2025 Quiz 7 Feingold

Show all work needed to justify your answers.

Carry out the following steps to diagonalize $A = \begin{bmatrix} 12 & -5 \\ 30 & -13 \end{bmatrix}$.

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(1) (2 Pts) Find $\det(A - tI_2) = \det \begin{bmatrix} 12-t & -5 \\ 30 & -13-t \end{bmatrix}$ and write it in factored form $(t-\lambda_1)(t-\lambda_2)$ to get the eigenvalues λ_1 and λ_2.

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(2) (2 Pts) Find a basis vector w_1 for the λ_1-eigenspace by solving $[A - \lambda_1 I_2|0]$.

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(3) (2 Pts) Find a basis vector w_2 for the λ_2-eigenspace by solving $[A - \lambda_2 I_2|0]$.

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(4) (2 Pts) Find the invertible matrix P whose columns are w_1 and w_2, and find P^{-1}.

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(5) (2 Pts) Compute $P^{-1}AP$ and verify that it is the diagonal matrix D with diagonal entries λ_1 and λ_2.

Math 304-6 Linear Algebra Spring 2025 Quiz 7 Solutions Feingold

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Carry out the following steps to diagonalize $A = \begin{bmatrix} 12 & -5 \\ 30 & -13 \end{bmatrix}$.

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(1) (2 Pts) $\det(A - tI_2) = \det \begin{bmatrix} 12-t & -5 \\ 30 & -13-t \end{bmatrix} = (12-t)(-13-t) - (-5)(30) = t^2 + t - 156 + 150 = t^2 + t - 6 = (t-2)(t+3)$ so the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -3$.

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(2) (2 Pts) The λ_1-eigenspace is found by solving $[A - 2I_2|0]$. Row reduce

$$\left[\begin{array}{cc|c} 10 & -5 & 0 \\ 30 & -15 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ so } 2x_1 = x_2 \text{ has basis vector } w_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

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(3) (2 Pts) The λ_2-eigenspace is found by solving $[A + 3I_2|0]$. Row reduce

$$\left[\begin{array}{cc|c} 15 & -5 & 0 \\ 30 & -10 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ so } 3x_1 = x_2 \text{ has basis vector } w_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

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(4) (2 Pts) The invertible matrix $P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

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(5) (2 Pts)

$$P^{-1}AP = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 12 & -5 \\ 30 & -13 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

is the diagonal matrix D with diagonal entries λ_1 and λ_2.
