NAME (Printed): ____

Math 304-6 Linear Algebra Spring 2025 Quiz 9 Feingold

INSTRUCTIONS: Show all calculations and reasons needed to justify your answers.

Let $V = \mathbb{R}^4$ with the standard dot product. Let $W = \langle T \rangle$ where

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is an ordered list.}$$

(1) (6 Pts) Use the Gram-Schmidt process to convert T into an **orthogonal** basis $T' = \{w_1' = w_1, w_2', w_3'\}$ for W. **Please rescale your answers to avoid fractions**.

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(2) (4 Pts) Use your answer to part (1) to find the coefficients, x_1, x_2, x_3, of the projection, $Proj_W(v) = \sum_{i=1}^{3} x_i w_i'$ of the vector $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in V$ into the subspace W. They are uniquely determined by the condition that $v - Proj_W(v)$ is orthogonal to W, that is, $(v - Proj_W(v)) \cdot w_j' = 0$ for $1 \leq j \leq 3$.

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Let $V = \mathbb{R}^4$ with the standard dot product. Let $W = \langle T \rangle$ where

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is an ordered list.}$$

(1) (6 Pts) Use the Gram-Schmidt process to convert T into an **orthogonal** basis $T' = \{w_1' = w_1, w_2', w_3'\}$ for W. **Answers are rescaled to avoid fractions**.

Solution: Step 1: $w_1' = w_1$.

Step 2: $w_2' = w_2 - \frac{w_2 \cdot w_1'}{w_1' \cdot w_1'} w_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{10}{30} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ which we rescale to $\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

Step 3: $$w_3' = w_3 - \frac{w_3 \cdot w_1'}{w_1' \cdot w_1'} w_1' - \frac{w_3 \cdot w_2'}{w_2' \cdot w_2'} w_2' = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{30} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}.$$

So, rescaling to avoid fractions, $$T' = \left\{ w_1' = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, w_2' = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, w_3' = \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

Check that $w_i' \cdot w_j' = 0$ for $1 \leq i < j \leq 3$, and by the process, $\langle T' \rangle = \langle T \rangle$.

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(2) (4 Pts) Use your answer to part (1) to find the coefficients, x_1, x_2, x_3, of the projection, $Proj_W(v) = \sum_{i=1}^{3} x_i w_i'$ of the vector $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in V$ into the subspace W. They are uniquely determined by the condition that $v - Proj_W(v)$ is orthogonal to W, that is, $(v - Proj_W(v)) \cdot w_j' = 0$ for $1 \leq j \leq 3$.

Solution: The conditions mean that $v \cdot w_j' = Proj_W(v) \cdot w_j' = x_j (w_j' \cdot w_j')$ for $1 \leq j \leq 3$ since T' is an orthogonal set. This says $x_j = \frac{v \cdot w_j'}{w_j' \cdot w_j'}$ so from part (1),

$$x_1 = \frac{a + 2b + 3c + 4d}{30}, \quad x_2 = \frac{2a + b - d}{6}, \quad x_3 = \frac{a - 3b + 3c - d}{20}.$$
