

SHOW ALL NECESSARY WORK. Note: A^T means the transpose of A .

(1) (10 Points, 2 points each) Answer each question separately. **Only justify part (b).**

- (a) If $X, Y \in \mathbb{R}^n$ and $\theta_{X,Y}$ is the angle between X and Y , what is the formula for $\cos(\theta_{X,Y})$ in terms of the standard dot product on \mathbb{R}^n ?
 - (b) Let $L : V \rightarrow V$ be **invertible** and suppose $v \in V$ is an eigenvector for L with eigenvalue $0 \neq \lambda \in \mathbb{F}$. Prove that v is an eigenvector for L^{-1} with eigenvalue λ^{-1} . Do not assume L is diagonalizable.
 - (c) Suppose $A \in \mathbb{F}_n^n$ and $Char_A(\lambda) = \det(\lambda I_n - A) = (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r}$ with r distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ in \mathbb{F} . Let T_i be a basis of eigenspace A_{λ_i} . What is the **most** you can say in general about the union $T = T_1 \cup \cdots \cup T_r$?
 - (d) With notation as in part (c), what property of set T means that A is diagonalizable?
 - (e) Let $E \in \mathbb{F}_n^n$ be an elementary matrix corresponding to an elementary **adder** row operation. What can you say about $\det(E)$?
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(2) (10 Points, 2 points each) Answer each question separately. **Show all work.**

- (a) Find $\det \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix}$. See the hint for problem 4(a).
 - (b) If $A \in \mathbb{F}_n^n$ has $\det(A) = 5$ what is $\det(3A^{-1})$?
 - (c) If $A^T = -A$ for $A \in \mathbb{F}_n^n$ where n is **odd**, what is the **most** you can say about $\det(A)$?
 - (d) Suppose $A, B \in \mathbb{F}_n^n$ are **similar**, that is, $B = P^{-1}AP$ for some invertible $P \in \mathbb{F}_n^n$. What is the relationship between $\det(A)$ and $\det(B)$?
 - (e) Suppose $A = A^T \in \mathbb{R}_n^n$ and $\lambda \neq \mu$ in \mathbb{R} are eigenvalues of A with corresponding eigenvectors, X and Y , so $AX = \lambda X$ and $AY = \mu Y$. What is the **most** you can say about $X \cdot Y$?
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(3) (10 Points, 2 points each) Let V be a real inner product space with inner product (\cdot, \cdot) . Answer each question separately.

- (a) State the Cauchy-Schwarz inequality for V .
 - (b) If $T = \{v_1, \dots, v_m\}$ is an **orthogonal** set of nonzero vectors in V , what is the most you can be sure about T ?
 - (c) Let $S = \{v_1, \dots, v_n\}$ be an **orthogonal basis** of V , so for any $v \in V$ we can write $v = \sum_{j=1}^n c_j v_j$. Use the inner product to give a formula for the coefficients c_j .
 - (d) Let $T = \{v_1, \dots, v_m\}$ be an **orthogonal basis** for a subspace $W \leq V$. Write the formula for the **projection** $Proj_W(v)$ of any vector $v \in V$ into W .
 - (e) State the definition of **positive definite** for an inner product (\cdot, \cdot) .
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(4) (15 Points) Let $A = \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \in \mathbb{R}_4^4$.

- (a) (4 Pts) Find the **characteristic polynomial** of A , $\det(A - tI_4)$, find all **eigenvalues**, $\lambda_i \in \mathbb{R}$, of A and the corresponding **algebraic multiplicities**, k_i . **Hint:** Using the last row of $A - tI_4$, adder row operations simplify the first three rows so that linear factors come out of $\det(A - tI_4)$.
- (b) (6 Pts) For each eigenvalue, λ_i , of A , find a **basis** for the **eigenspace**, A_{λ_i} , and the **geometric multiplicity** $g_i = \dim(A_{\lambda_i})$.
- (c) (5 Pts) Determine whether or not A is **diagonalizable** over \mathbb{R} . If it is, find real matrices D and P such that $D = P^{-1}AP$ is diagonal. If not, explain why.

(5) (20 Points) Let $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$ and $T = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be

an ordered basis for a subspace W of \mathbb{R}^4 .

- (a) Use the Gram-Schmidt orthogonalization process to convert T into an **orthogonal** basis of W , $T' = \{w'_1, w'_2, w'_3\}$, where

$$w'_1 = w_1, \quad w'_2 = w_2 - Proj_{\langle w'_1 \rangle}(w_2) \quad \text{and} \quad w'_3 = w_3 - Proj_{\langle w'_1, w'_2 \rangle}(w_3).$$

After using these formulas, **please rescale** w'_i if necessary to **avoid fractions**. Be sure to check your work by verifying that $w'_i \cdot w'_j = 0$ for all $1 \leq i < j \leq 3$.

- (b) Use your answer to part (a) to **find the projection**, $Proj_W(v) = \sum_{i=1}^3 x_i w'_i$ of a general vector v into the subspace W . The **first step** is to find the coefficients, x_1, x_2, x_3 , which are uniquely determined by the condition that $v - Proj_W(v)$ is orthogonal to W , that is, $(v - Proj_W(v)) \cdot w'_j = 0$ for $1 \leq j \leq 3$. After you have found the coefficients, **write the formula** for $Proj_W(v)$ in terms of a, b, c, d and **simplify all expressions** to get your final answer. **Hint:** To check your simplified formula, see if it satisfies $Proj_W(w_j) = w_j$ for $1 \leq j \leq 3$.

- (6) (10 Points) Let $W = \langle T \rangle$ from problem (5).

- (a) (5 Pts) Find a **basis** for $W^\perp = \{X \in \mathbb{R}^4 \mid w_i \cdot X = 0, 1 \leq i \leq 3\}$.
- (b) (5 Pts) Show that $W + W^\perp = \mathbb{R}^4$ is an **orthogonal direct sum**.

(1) (10 Points, 2 points each) Answer each question separately. **Only justify part (b).**

(a) If $X, Y \in \mathbb{R}^n$ and $\theta_{X,Y}$ is the angle between X and Y , the formula is

$$\cos(\theta_{X,Y}) = \frac{X \cdot Y}{\sqrt{X \cdot X} \sqrt{Y \cdot Y}} \text{ since } \|X\| = \sqrt{X \cdot X}.$$

(b) Let $L : V \rightarrow V$ be **invertible** and suppose $v \in V$ is an eigenvector for L with eigenvalue $0 \neq \lambda \in \mathbb{F}$. Prove that v is an eigenvector for L^{-1} with eigenvalue λ^{-1} . Do not assume L is diagonalizable. We know $v = L^{-1}(L(v)) = L^{-1}(\lambda v) = \lambda L^{-1}(v)$ so $\lambda^{-1}v = L^{-1}(v)$ shows that v is an eigenvector for L^{-1} with eigenvalue λ^{-1} .

(c) Suppose $A \in \mathbb{F}_n^n$ and $\text{Char}_A(\lambda) = \det(\lambda I_n - A) = (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \dots (\lambda - \lambda_r)^{k_r}$ with r distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ in \mathbb{F} . Let T_i be a basis of eigenspace A_{λ_i} . The **most** you can say in general about the union $T = T_1 \cup \dots \cup T_r$ is that it is an **independent** set of $g_1 + \dots + g_r$ eigenvectors.

(d) With notation as in part (c), the property that T is a basis of \mathbb{F}^n means that A is diagonalizable. Other answers are: T contains n vectors, or T spans \mathbb{F}^n .

(e) Let $E \in \mathbb{F}_n^n$ be an elementary matrix corresponding to an elementary **adder** row operation. Then $\det(E) = 1$.

(2) (10 Points, 2 points each) Answer each question separately. **Show all work.**

$$\begin{aligned} \text{(a) } \det \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} &= \det \begin{bmatrix} -2 & 0 & 0 & 8 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 2 \\ 5 & -5 & -5 & -7 \end{bmatrix} = \\ &(-2)^3 \det \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 5 & -5 & -5 & -7 \end{bmatrix} = (-2)^3 \det \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = -24. \end{aligned}$$

(b) If $A \in \mathbb{F}_n^n$ has $\det(A) = 5$ then $\det(3A^{-1}) = 3^n \det(A^{-1}) = \frac{3^n}{5}$.

(c) If $A^T = -A$ for $A \in \mathbb{F}_n^n$ where n is **odd**, then $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$ since n is odd, so $\det(A) = 0$.

(d) Suppose $A, B \in \mathbb{F}_n^n$ are **similar**, that is, $B = P^{-1}AP$ for some invertible $P \in \mathbb{F}_n^n$. Then $\det(A) = \det(B)$ since $\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \det(A) \det(P) = \det(P)^{-1} \det(A) \det(P) = \det(A)$.

(e) Suppose $A = A^T \in \mathbb{R}_n^n$ and $\lambda \neq \mu$ in \mathbb{R} are eigenvalues of A with corresponding eigenvectors, X and Y , so $AX = \lambda X$ and $AY = \mu Y$. We can say $X \cdot Y = 0$ was proved in class as follows. $\lambda(X \cdot Y) = (\lambda X) \cdot Y = (AX) \cdot Y = X \cdot (A^T Y) = X \cdot (AY) = X \cdot (\mu Y) = \mu(X \cdot Y)$ so $(\lambda - \mu)(X \cdot Y) = 0$ and since $\lambda \neq \mu$, we get $X \cdot Y = 0$.

(3) (10 Points, 2 points each) Let V be a real inner product space with inner product (\cdot, \cdot) . Answer each question separately.

(a) State the Cauchy-Schwarz inequality for V .

Solution: For any $u, v \in V$ we have $(u, v)^2 \leq (u, u)(v, v)$.

(b) If $T = \{v_1, \dots, v_m\}$ is an **orthogonal** set of nonzero vectors in V , what is the most you can be sure about T ?

Solution: You can be sure that T is independent.

(c) Let $S = \{v_1, \dots, v_n\}$ be an **orthogonal basis** of V , so for any $v \in V$ we can write $v = \sum_{j=1}^n c_j v_j$. Use the inner product to give a formula for the coefficients c_j .

Solution: The formula is $c_j = \frac{(v, v_j)}{(v_j, v_j)}$.

(d) Let $T = \{v_1, \dots, v_m\}$ be an **orthogonal basis** for a subspace $W \leq V$. Write the formula for the **projection** $Proj_W(v)$ of any vector $v \in V$ into W .

Solution: The formula for the projection is $Proj_W(v) = \sum_{i=1}^m \frac{(v, v_i)}{(v_i, v_i)} v_i$.

(e) State the definition of **positive definite** for an inner product (\cdot, \cdot) .

Solution: The symmetric bilinear form (\cdot, \cdot) is positive definite when for all $v \in V$, $(v, v) \geq 0$ and $(v, v) = 0$ iff $v = \theta$ is the zero vector.

(4) (15 Points) Let $A = \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \in \mathbb{R}_4^4$.

(a) (4 Points) The characteristic polynomial is $Char_A(t) = \det(tI_4 - A) = \det(A - tI_4)$

$$\begin{aligned} \det \begin{bmatrix} 18-t & -20 & -20 & -20 \\ 5 & -7-t & -5 & -5 \\ 5 & -5 & -7-t & -5 \\ 5 & -5 & -5 & -7-t \end{bmatrix} &= \det \begin{bmatrix} -t-2 & 0 & 0 & 4t+8 \\ 0 & -t-2 & 0 & t+2 \\ 0 & 0 & -t-2 & t+2 \\ 5 & -5 & -5 & -7-t \end{bmatrix} \\ &= (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 5 & -5 & -5 & -7-t \end{bmatrix} = (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -5 & -5 & -t+13 \end{bmatrix} \\ &= (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -5 & -t+8 \end{bmatrix} = (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -t+3 \end{bmatrix} \\ &= (t+2)^3 (t-3). \quad (\text{Constant term, } Char_A(0) = -24 = \det(A) \text{ answers problem 2(a).}) \end{aligned}$$

So the eigenvalues are $\lambda_1 = -2$ with algebraic multiplicity $k_1 = 3$ and $\lambda_2 = 3$ with algebraic multiplicity $k_2 = 1$.

(b) (6 Points) Check the $\lambda_1 = -2$ eigenspace first since the algebraic multiplicity $k_1 = 3$. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $\lambda = -2$ to $A - \lambda I_4$. Row reduce

$$\left[\begin{array}{cccc|c} 20 & -20 & -20 & -20 & 0 \\ 5 & -5 & -5 & -5 & 0 \\ 5 & -5 & -5 & -5 & 0 \\ 5 & -5 & -5 & -5 & 0 \end{array} \right] \text{ to } \left[\begin{array}{cccc|c} 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ so } \begin{matrix} x_1 = r + s + t \\ x_2 = r \in \mathbb{R} \\ x_3 = s \in \mathbb{R} \\ x_4 = t \in \mathbb{R} \end{matrix}, \text{ then}$$

$$A_{\lambda_1} = \left\{ \left[\begin{array}{c} r+s+t \\ r \\ s \\ t \end{array} \right] \in \mathbb{R}^4 \mid r, s, t \in \mathbb{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ so } g_1 = 3.$$

Since there will be one more independent eigenvector from the other eigenvalue, we will have the necessary four eigenvectors to form a basis for \mathbb{R}^4 , so this A is diagonalizable.

Now find the $\lambda_2 = 3$ eigenspace. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $\lambda = 3$ to $A - \lambda I_4$. Row reduce

$$\left[\begin{array}{cccc|c} 15 & -20 & -20 & -20 & 0 \\ 5 & -10 & -5 & -5 & 0 \\ 5 & -5 & -10 & -5 & 0 \\ 5 & -5 & -5 & -10 & 0 \end{array} \right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ so } \begin{matrix} x_1 = 4r \\ x_2 = r \\ x_3 = r \\ x_4 = r \in \mathbb{R} \end{matrix}, \text{ then}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} 4r \\ r \\ r \\ r \end{bmatrix} \in \mathbb{R}^4 \mid r \in \mathbb{R} \right\} \quad \text{has basis} \quad \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{so } g_2 = 1.$$

(c) (5 Points) Therefore, $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, ${}_T D_T = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

and

$$P = {}_S P_T = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad P^{-1} = {}_T P_S = \begin{bmatrix} -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix}. \quad \text{Check:}$$

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -4 & -2 & -2 \\ 2 & -2 & -4 & -2 \\ 2 & -2 & -2 & -4 \\ 3 & -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = {}_T D_T. \end{aligned}$$

(5) (20 Points) The set $T = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an ordered basis for a subspace W of \mathbb{R}^4 .

(a) Use the Gram-Schmidt orthogonalization process to convert T into an **orthogonal** basis T' for W .

Solution: (10 Pts) First let $w'_1 = w_1$, then

$$w'_2 = w_2 - \frac{w_2 \cdot w'_1}{w'_1 \cdot w'_1} w'_1 = w_2 - \frac{12}{4} w'_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and check that } w'_1 \cdot w'_2 = 0.$$

Let

$$w'_3 = w_3 - \frac{w_3 \cdot w'_1}{w'_1 \cdot w'_1} w'_1 - \frac{w_3 \cdot w'_2}{w'_2 \cdot w'_2} w'_2 = w_3 - \frac{3}{4} w'_1 - \frac{1}{4} w'_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

We may double this vector and it will still be orthogonal to the previous two vectors, so

$$T' = \left\{ w'_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w'_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, w'_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is an orthogonal basis for W obtained by the Gram-Schmidt orthogonalization process.

- (b) Use your answer to part (a) to **find the projection**, $Proj_W(v) = \sum_{i=1}^3 x_i w'_i$ of a general vector v into the subspace W . The coefficients, x_1, x_2, x_3 , are uniquely determined by the condition that $v - Proj_W(v)$ is orthogonal to W , that is, $(v - Proj_W(v)) \cdot w'_j = 0$ for $1 \leq j \leq 3$. After you have found the coefficients, write the formula for $Proj_W(v)$ in terms of a, b, c, d .

Solution: (10 Pts) The conditions mean that $v \cdot w'_j = Proj_W(v) \cdot w'_j = x_j(w'_j \cdot w'_j)$ for $1 \leq j \leq 3$ since T' is an orthogonal set. This says $x_j = \frac{v \cdot w'_j}{w'_j \cdot w'_j}$ so from part (a),

$$x_1 = \frac{a + b + c + d}{4}, \quad x_2 = \frac{-a + b - c + d}{4}, \quad x_3 = \frac{-a + c}{2} \quad \text{so}$$

$$\begin{aligned} Proj_W(v) &= \frac{(a + b + c + d)}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{(-a + b - c + d)}{4} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{(-a + c)}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} (a + b + c + d) - (-a + b - c + d) - (-2a + 2c) \\ (a + b + c + d) + (-a + b - c + d) \\ (a + b + c + d) - (-a + b - c + d) + (-2a + 2c) \\ (a + b + c + d) + (-a + b - c + d) \end{bmatrix} = \begin{bmatrix} a \\ (b + d)/2 \\ c \\ (b + d)/2 \end{bmatrix}. \end{aligned}$$

This formula does satisfy $Proj_W(w_j) = w_j$ for $1 \leq j \leq 3$.

(6) (10 Points) Let $W = \langle T \rangle$ from problem (5).

(a) (5 Pts) Find a **basis** for $W^\perp = \{X \in \mathbb{R}^4 \mid w_i \cdot X = 0, 1 \leq i \leq 3\}$.

Solution: (5 Pts) Find W^\perp by solving $[A|0_1^3]$ where $Row_i(A) = w_i^T$. Row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 4 & 2 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array} \right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] \text{ so } \begin{array}{l} x_1 = 0 \\ x_2 = -r \\ x_3 = 0 \\ x_4 = r \in \mathbb{R} \end{array} \text{ so } W^\perp \text{ has basis } \left\{ w_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) (5 Pts) Show that $W + W^\perp = \mathbb{R}^4$ is an **orthogonal direct sum**.

Solution: (5 Pts) $T \cup \{w_4\}$ is a basis of \mathbb{R}^4 since T is independent and $w_4 \notin \langle T \rangle$. Also, $w_4 \perp T$, so the sum is orthogonal, and therefore, direct. In fact, $W \cap W^\perp = \{0_1^4\}$ because only the zero vector is orthogonal to itself.
