NAME (Printed): ______________________________

Math 304-6 Linear Algebra Spring 2026 Quiz 4 Feingold

INSTRUCTIONS: Show all calculations needed to justify your answers.

(1) (5 Pts) Let m, n, p be positive integers and let $A \in \mathbb{F}_n^m$ be an $m \times n$ matrix fixed for this problem. Then A determines a set of $n \times p$ matrices $W = \left\{ B \in \mathbb{F}_p^n \mid AB = 0_p^m \right\}$. Show W is a subspace of $\mathbb{F}_p^n$.

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(2) (5 Pts) Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \in \mathbb{F}_2^2$. Find $W = \left\{ B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}_2^2 \;\middle|\; AB = 0_2^2 \right\}$ by solving a linear system.

Math 304-6 Linear Algebra Spring 2026 Quiz 4 Solutions Feingold

INSTRUCTIONS: Show all calculations needed to justify your answers.

(1) (5 Pts) Let m, n, p be positive integers and let $A \in \mathbb{F}^m_n$ be an $m \times n$ matrix fixed for this problem. Then A determines a set of $n \times p$ matrices $W = \left\{B \in \mathbb{F}^n_p \mid AB = 0^m_p\right\}$. Show W is a subspace of $\mathbb{F}^n_p$.

SOLUTION 1: First show W is closed under addition. Let $B_1, B_2 \in W$, so $AB_1 = 0^m_p$ and $AB_2 = 0^m_p$. Then $A(B_1 + B_2) = AB_1 + AB_2 = 0^m_p + 0^m_p = 0^m_p$ means $B_1 + B_2 \in W$.

Next show W is closed under scalar multiplication. Let $B \in W$ so $AB = 0^m_p$, and let $r \in \mathbb{F}$. Then $A(rB) = r(AB) = r0^m_p = 0^m_p$ means $rB \in W$.

Finally, $0^n_p \in W$ since $A0^n_p = 0^m_p$.

SOLUTION 2: First show that the map $L : \mathbb{F}^n_p \to \mathbb{F}^m_p$ defined by $L(B) = AB$ is linear. For any $B_1, B_2 \in \mathbb{F}^n_p$ we have $L(B_1 + B_2) = A(B_1 + B_2) = AB_1 + AB_2 = L(B_1) + L(B_2)$. For any $B \in \mathbb{F}^n_p$ and any $r \in \mathbb{F}$ we have $L(rB) = A(rB) = r(AB) = rL(B)$. These are just properties of matrix operations, including that for $A \in \mathbb{F}^m_n$ and $B \in \mathbb{F}^n_p$ we have $AB \in \mathbb{F}^m_p$. Finally, $W = ker(L)$ is a subspace of $\mathbb{F}^n_p$ since the kernel of any linear map is a subspace of its domain.

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(2) (5 Pts) Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \in \mathbb{F}^2_2$. Find $W = \left\{ B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^2_2 \;\middle|\; AB = 0^2_2 \right\}$ by solving a linear system.

SOLUTION: To find all $B \in \mathbb{F}^2_2$ such that $AB = 0^2_2$, write out the condition as

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (a+3c) & (b+3d) \\ (3a+9c) & (3b+9d) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives us four equations in four variables which we solve by row reduction as follows:

$$\text{Row reduce } \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 3 & 0 & 9 & 0 & 0 \\ 0 & 3 & 0 & 9 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} a = -3c \\ b = -3d \\ c \in \mathbb{F} \text{ free} \\ d \in \mathbb{F} \text{ free} \end{array}$$

so $W = \left\{ B = \begin{bmatrix} -3c & -3d \\ c & d \end{bmatrix} \in \mathbb{F}^2_2 \;\middle|\; c, d \in \mathbb{F} \right\}$.
