NAME (Printed): ____

Math 304-6 Linear Algebra Spring 2026 Quiz 9 Feingold

INSTRUCTIONS: Show all calculations and reasons needed to justify your answers.

Let $V = \mathbb{R}^4$ with the standard dot product. Let $W = \langle T \rangle$ where

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is an ordered list.}$$

(1) (6 Pts) Use the Gram-Schmidt process to convert T into an **orthogonal** basis $T' = \{w'_1 = w_1, w'_2, w'_3\}$ for W. **Please rescale your answers to avoid fractions**.

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(2) (4 Pts) Use your answer to part (1) to find the coefficients, x_1, x_2, x_3, of the projection, $Proj_W(v) = \sum_{i=1}^{3} x_i w'_i$ of the vector $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in V$ into the subspace W. They are uniquely determined by the condition that $v - Proj_W(v)$ is orthogonal to W, that is, $(v - Proj_W(v)) \cdot w'_j = 0$ for $1 \leq j \leq 3$.

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Let $V = \mathbb{R}^4$ with the standard dot product. Let $W = \langle T \rangle$ where

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is an ordered list.}$$

(1) (6 Pts) Use the Gram-Schmidt process to convert T into an **orthogonal** basis $T' = \{w'_1 = w_1, w'_2, w'_3\}$ for W. **Answers are rescaled to avoid fractions**.

Solution: Step 1: $w'_1 = w_1$.

Step 2: $w'_2 = w_2 - \frac{w_2 \cdot w'_1}{w'_1 \cdot w'_1} w'_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{10}{30} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ which we rescale to $\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

$$\text{Step 3:} \quad w'_3 = w_3 - \frac{w_3 \cdot w'_1}{w'_1 \cdot w'_1} w'_1 - \frac{w_3 \cdot w'_2}{w'_2 \cdot w'_2} w'_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{30} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}.$$

So, rescaling to avoid fractions, $T' = \left\{ w'_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, w'_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, w'_3 = \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} \right\}$.

Check that $w'_i \cdot w'_j = 0$ for $1 \leq i < j \leq 3$, and by the process, $\langle T' \rangle = \langle T \rangle$.

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(2) (4 Pts) Use your answer to part (1) to find the coefficients, x_1, x_2, x_3, of the projection, $Proj_W(v) = \sum_{i=1}^{3} x_i w'_i$ of the vector $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in V$ into the subspace W. They are uniquely determined by the condition that $v - Proj_W(v)$ is orthogonal to W, that is, $(v - Proj_W(v)) \cdot w'_j = 0$ for $1 \leq j \leq 3$.

Solution: The conditions mean that $v \cdot w'_j = Proj_W(v) \cdot w'_j = x_j (w'_j \cdot w'_j)$ for $1 \leq j \leq 3$ since T' is an orthogonal set. This says $x_j = \frac{v \cdot w'_j}{w'_j \cdot w'_j}$ so from part (1),

$$x_1 = \frac{a + 2b + 3c + 4d}{30}, \quad x_2 = \frac{2a + b - d}{6}, \quad x_3 = \frac{a - 3b + 3c - d}{20}.$$
