Math 304 Linear Algebra Fall 2012 Exam 3 Feingold

(1) (30 Points, 5 points each) Answer each question separately.

(a) If $A \in \mathbf{R}_n^n$ and the homogeneous linear system $[A|0]$ has non-trivial solutions, then what is the most you can say about $det(A)$?

(b) Suppose $L : V \to V$ and $v \in V$ is an eigenvector for L with eigenvalue $\lambda \in \mathbf{R}$. Show v is also an eigenvector for $L^2 = L \circ L$ with eigenvalue λ^2.

(c) Let $dim(V) = n$ and suppose $L : V \to V$ has characteristic polynomial $p_L(t) = (-1)^n (t - \lambda_1)^{k_1} (t - \lambda_2)^{k_2} \cdots (t - \lambda_r)^{k_r}$ with r distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$. What is the most you can say about the sum of all the algebraic multiplicities, $k_1 + k_2 + \cdots + k_r$?

(d) With notation as in part (c), let $T_i = \{v_{ij} \mid 1 \le j \le g_i\}$ be a **basis** of the λ_i eigenspace, L_{λ_i}. What is the **most** you can say about the union of all these sets $T = T_1 \cup \cdots \cup T_r$?

(e) Let $E \in \mathbf{R}_n^n$ be the elementary matrix corresponding to an elementary adder row operation, let $A \in \mathbf{R}_n^n$ and let $B = EA$ be the matrix obtained by doing the adder row operation to A. What is the most you can say about $\det(B)$?

(f) Let $S = \{v_1 = [1 \quad 1 \quad 1 \quad 1]^T, v_2 = [1 \quad 2 \quad 3 \quad 4]^T\} \subset \mathbf{R}^4$. Find $S^\perp$.

(2) (30 points, 5 points each) Answer each question separately.

(a) Find $\det \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix}$.

(b) If $A \in \mathbf{R}_n^n$ has $\det(A) = 0$ then what is the most you can say about $rank(A)$?

(c) If $A \in \mathbf{R}_n^n$ and there is a non-trivial dependence relation among the columns of A, what is the most you can say about $\det(A)$?

(d) Let the columns of $A \in \mathbf{R}_n^n$ form an orthonormal basis of $\mathbf{R}^n$. What is the most you can say about $\det(A)$?

(e) If $A^T = -A$ for a matrix $A \in \mathbf{R}_n^n$ where n is **odd**, what is the **most** you can say about $det(A)$?

(f) Let $A, B, C \in \mathbf{R}_n^n$ with $det(A) = 3$, $det(B) = -7$ and $\det(C) = 2$. Find $det(A^T B^{-1} C^3)$.

(3) (40 Pts) For each $A \in \mathbf{R}_n^n$ given below define $L : \mathbf{R}^n \to \mathbf{R}^n$ by $L(X) = AX$ so $A = {}_S[L]_S$ is the matrix representing L with respect to the standard basis S of $\mathbf{R}^n$. Answer the following questions.

(1) Find the **characteristic polynomial** of L, $p_A(t) = \det(A - tI_n)$, find the **eigenvalues** of L (the roots of $p_A(t)$), and their **algebraic multiplicities**.

(2) Can L can be diagonalized? **If not, give reasons why**. If it can, **find a basis** T of $\mathbf{R}^n$ and a **diagonal matrix** $D = {}_T[L]_T$ representing L from T to T. Also **find the transition matrix** $P = {}_S P_T$ and check that $D = P^{-1}AP$.

$$(a) \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \qquad (b) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

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(1) (30 Points, 5 points each) Answer each question separately.

(a) If $[A|0]$ has non-trivial solutions then A row reduces to a matrix with a zero row, A is not invertible, so $\det(A) = 0$.

(b) Since $L(v) = \lambda v$ we have $L^2(v) = (L \circ L)(v) = L(L(v)) = L(\lambda v) = \lambda L(v) = \lambda\lambda v = \lambda^2 v$ shows v is an eigenvector for L^2 with eigenvalue λ^2.

(c) We can say that $k_1 + k_2 + \cdots + k_r = n$.

(d) $T = T_1 \cup \cdots \cup T_r$ is **linearly independent**.

(e) $B = EA$ so $\det(B) = \det(EA) = \det(E)\det(A) = (1)\det(A)$ since $\det(E) = 1$ for an adder type E.

(f) Let $S = \{v_1 = [1 \quad 1 \quad 1 \quad 1]^T, v_2 = [1 \quad 2 \quad 3 \quad 4]^T\} \subset \mathbf{R}^4$. Find $S^\perp = \{X \in \mathbf{R}^4 \mid v_1 \cdot X = 0 = v_2 \cdot X\}$. This is the solution space of $[A|0]$ where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Since A row reduces to $A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ the solution space $S^\perp = \{[\,(r+2s) \quad (-2r-3s) \quad r \quad s\,]^T \in \mathbf{R}^4 \mid r, s \in \mathbf{R}\}$.

(2) (30 Points, 5 Points each)

(a) $$\det \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} = \det \begin{bmatrix} -2 & 0 & 0 & 8 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 2 \\ 5 & -5 & -5 & -7 \end{bmatrix} =$$
$$(-2)^3 \det \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 5 & -5 & -5 & -7 \end{bmatrix} = (-2)^3 \det \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = -24.$$

(b) $\det(A) = 0$ so A is not invertible, so $rank(A) < n$.

(c) If $A \in \mathbf{R}^n_n$ and there is a non-trivial dependence relation among the columns of A, then $\det(A) = 0$.

(d) Since the columns of $A \in \mathbf{R}^n_n$ form an orthonormal basis of $\mathbf{R}^n$, we know $I_n = A^T A$ so $1 = \det(I_n) = \det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A) = \det(A)^2$, so $\det(A) = \pm 1$.

(e) If $A^T = -A$ for $A \in \mathbf{R}^n_n$ where n is odd, then $det(A) = det(A^T) = det(-A) = (-1)^n det(A) = -det(A)$, taking out a factor of -1 from each row, so we can say that $det(A) = 0$.

(f) $$det(A^T B^{-1} C^3) = \frac{det(A)det(C)^3}{det(B)} = \frac{(3)(2^3)}{-7} = \frac{24}{-7}.$$

(3) (40 Points)
(a) (30 Points) (1) The characteristic polynomial is $p_A(\lambda) = \det(A - \lambda I_4) =$

$$\det \begin{bmatrix} 18-\lambda & -20 & -20 & -20 \\ 5 & -7-\lambda & -5 & -5 \\ 5 & -5 & -7-\lambda & -5 \\ 5 & -5 & -5 & -7-\lambda \end{bmatrix} = \det \begin{bmatrix} -\lambda-2 & 0 & 0 & 4\lambda+8 \\ 0 & -\lambda-2 & 0 & \lambda+2 \\ 0 & 0 & -\lambda-2 & \lambda+2 \\ 5 & -5 & -5 & -7-\lambda \end{bmatrix}$$

$$= (\lambda+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 5 & -5 & -5 & -7-\lambda \end{bmatrix} = (\lambda+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -5 & -5 & -\lambda+13 \end{bmatrix}$$

$$= (\lambda+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -5 & -\lambda+8 \end{bmatrix} = (\lambda+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -\lambda+3 \end{bmatrix}$$

$= (\lambda+2)^3\ (\lambda-3)$. (Constant term, $p_A(0) = -24 = \det(A)$ answers problem 2(a).

So the eigenvalues are $\lambda_1 = -2$ with algebraic multiplicity $k_1 = 3$ and $\lambda_2 = 3$ with algebraic multiplicity $k_2 = 1$.

(2) Check the $\lambda_1 = -2$ eigenspace first since the algebraic multiplicity $k_1 = 3$. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $\lambda = -2$ to $A - \lambda I_4$. Row reduce

$$\left[\begin{array}{cccc|c} 20 & -20 & -20 & -20 & 0 \\ 5 & -5 & -5 & -5 & 0 \\ 5 & -5 & -5 & -5 & 0 \\ 5 & -5 & -5 & -5 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r+s+t \\ x_2 = r \in \mathbf{R} \\ x_3 = s \in \mathbf{R} \\ x_4 = t \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} r+s+t \\ r \\ s \\ t \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r,s,t \in \mathbf{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

with three vectors. Since there will be one more independent eigenvector from the other eigenvalue, we will have the necessary four eigenvectors to form a basis for $\mathbf{R}^4$, so this A is diagonalizable.

Now find the $\lambda_2 = 3$ eigenspace. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $\lambda = 3$ to $A - \lambda I_4$. Row reduce

$$\left[\begin{array}{cccc|c} 15 & -20 & -20 & -20 & 0 \\ 5 & -10 & -5 & -5 & 0 \\ 5 & -5 & -10 & -5 & 0 \\ 5 & -5 & -5 & -10 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 4r \\ x_2 = r \\ x_3 = r \\ x_4 = r \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} 4r \\ r \\ r \\ r \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r \in \mathbf{R} \right\} \qquad \text{has basis} \qquad \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Therefore, $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, ${}_TD_T = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and

$P = {}_SP_T = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = {}_TP_S = \begin{bmatrix} -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix}$. Check:

$$P^{-1}AP = \begin{bmatrix} -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & -2 & -2 \\ 2 & -2 & -4 & -2 \\ 2 & -2 & -2 & -4 \\ 3 & -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = {}_TD_T.$$

3(b) (10 Points) (1) The characteristic polynomial is $p_A(\lambda) = \det(A - \lambda I_3) =$

$$\det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (1-\lambda)^2(2-\lambda) = -(\lambda-1)^2(\lambda-2)$$

So the eigenvalues are $\lambda_1 = 1$ with algebraic multiplicity $k_1 = 2$ and $\lambda_2 = 2$ with algebraic multiplicity $k_2 = 1$.

(2) First check the $\lambda_1 = 1$ eigenspace. Row reduce

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r \in \mathbf{R} \\ x_2 = 0 \\ x_3 = 0 \end{array}$$

so the $\lambda_1 = 1$ eigenspace A_{λ_1} only has dimension $1 = g_1 < k_1 = 2$ which means this L is not diagonalizable.