Math 304 Linear Algebra Fall 2019 Practice Exam 3 Feingold

SHOW ALL NECESSARY WORK FOR EACH PROBLEM

(1) (30 Pts) Let $V = \mathbf{R}^4$ with standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and with the standard dot product. Let $T = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and let $W = \langle T \rangle$ be the span of T. For $u, v \in V$ let $\theta_{u,v}$ be the angle between u and v, so $\cos(\theta_{u,v}) = \frac{u \cdot v}{||u||\ ||v||}$.

(a) Find $\cos(\theta_{w_1,w_2})$, $\cos(\theta_{w_2,w_3})$ and $\cos(\theta_{w_1,w_3})$.
(b) Find $W^{\perp} = \{v \in V \mid v \perp W\}$.
(c) Why must the matrix $M = [u_i \cdot u_j] \in \mathbf{R}^3_3$ be **positive definite**?

(2) (15 Pts) Answer each question separately.
(a) If $A, P \in \mathbf{R}^n_n$ are **invertible** matrices and $P^{-1}AP = A^{-1}$, what is the **most** you can say about $\det(A)$?
(b) Suppose $T = \{w_1, \cdots, w_m\}$ is **any** basis of subspace W in $\mathbf{R}^n$ and $\hat{v} = Proj_W(v) = \sum_{i=1}^{m} x_i w_i$ is the projection of v into W. Write the dot product formulas which define $\hat{v}$ uniquely as the solution to a linear system.
(c) Suppose $V = C[a, b]$ is the **inner product space** of all continuous functions $f : [a, b] \to \mathbf{R}$ with inner product $\langle f, g \rangle = \int_a^b f(t)g(t)dt$. Write the statement of the Cauchy-Schwarz Inequality for any $f, g \in V$ in terms of integrals.

(3) (20 Pts) The set $S = \{1, t, t^2\}$ is the standard basis for the inner product space $\mathbf{P}_2[t]$ with inner product $\langle p, q \rangle = \int_0^1 p(t)q(t)dt$. Use the Gram-Schmidt process to convert S into an **orthogonal** basis for $\mathbf{P}_2[t]$, but **do not** bother to do the normalizing step.

(4) (15 Pts) Suppose $A, B \in \mathbf{R}^n_n$ are **orthogonal** matrices.

(a) Show that AB is also orthogonal.
(b) Show that A^{-1} is also orthogonal.
(c) Show that any power A^k is also orthogonal for any integer k.

(5) (20 Pts) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
(a) (8 Pts) Find the **characteristic polynomial** of A, $\det(A - \lambda I_4)$, find all **eigenvalues**, λ_i, of A and the corresponding **algebraic multiplicities**, k_i.
(b) (8 Pts) For each eigenvalue, λ_i, of A, find a **basis** for the **eigenspace**, A_{λ_i}, and the **geometric multiplicity** $g_i = \dim(A_{\lambda_i})$.
(c) (4 Pts) Determine whether or not A is **diagonalizable**. If it is, find D and P such that $D = P^{-1}AP$ is diagonal. If not, explain why.

(1) (30 Points) Let $V = \mathbf{R}^4$ with standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and with the standard dot product. Let $T = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and let $W = \langle T \rangle$ be the span of T. For $u, v \in V$ let $\theta_{u,v}$ be the angle between u and v, so $\cos(\theta_{u,v}) = \frac{u \cdot v}{||u||\ ||v||}$.

(a) Find $\cos(\theta_{w_1,w_2})$, $\cos(\theta_{w_2,w_3})$ and $\cos(\theta_{w_1,w_3})$.

Solution: We get $w_1 \cdot w_1 = 4$, $w_1 \cdot w_2 = 12$, $w_1 \cdot w_3 = 3$, $w_2 \cdot w_2 = 40$, $w_2 \cdot w_3 = 10$, and $w_3 \cdot w_3 = 3$. So $\cos(\theta_{w_1,w_2}) = \frac{12}{\sqrt{4}\sqrt{40}} = \frac{3}{\sqrt{10}}$, $\cos(\theta_{w_2,w_3}) = \frac{10}{\sqrt{40}\sqrt{3}} = \frac{5}{\sqrt{30}}$ and $\cos(\theta_{w_1,w_3}) = \frac{3}{\sqrt{4}\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$.

(b) Find $W^\perp = \{v \in V \mid v \perp W\}$.

Solution: $v \in W^\perp$ when $v \cdot w_i = 0$ for $i = 1, 2, 3$. If $v = [a \quad b \quad c \quad d]^T$ then $v \cdot w_1 = a + b + c + d = 0$, $v \cdot w_2 = 2a + 4b + 2c + 4d = 0$ and $v \cdot w_3 = b + c + d = 0$ so the conditions on v are a linear system $AX = 0_1^3$ where the rows of A are w_1^T, w_2^T and w_3^T. The solutions of that system are $a = 0$, $b = -d$, $c = 0$ and $d \in \mathbf{R}$ is free. We get that $W^\perp = \langle [0 \quad -1 \quad 0 \quad 1]^T \rangle$ is the line in $\mathbf{R}^4$ with basis vector $-\mathbf{e}_2 + \mathbf{e}_4$.

(c) Why must the matrix $M = [w_i \cdot w_j] \in \mathbf{R}_3^3$ be **positive definite**?

Solution: M is the matrix of dot products of the basis T of subspace W inside $\mathbf{R}^4$, so W is an inner product space whose inner product is the standard dot product which is positive definite. So M must be positive definite.

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(2) (15 Pts) Answer each question separately.

(a) If $A, P \in \mathbf{R}_n^n$ are **invertible** matrices and $P^{-1}AP = A^{-1}$, what is the **most** you can say about $\det(A)$?

Solution: $\det(P^{-1})\det(A)\det(P) = \det(P^{-1}AP) = \det(A^{-1}) = \frac{1}{\det(A)}$ so $\det(P^{-1})\det(P)\det(A) = \det(A) = \frac{1}{\det(A)}$ so $(\det(A))^2 = 1$ so $\det(A) = \pm 1$.

(b) Suppose $T = \{w_1, \cdots, w_m\}$ is **any** basis of subspace W in $\mathbf{R}^n$ and $\hat{v} = Proj_W(v) = \sum_{i=1}^m x_i w_i$ is the projection of v into W. Write the dot product formulas which define $\hat{v}$ uniquely as the solution to a linear system.

Solution: The formulas are $(\hat{v} - v) \cdot w_j = 0$ for $1 \le j \le m$, which means $\sum_{i=1}^m x_i w_i \cdot w_j = v \cdot w_j$. This is the linear system $AX = B$ where $A = [w_i \cdot w_j] \in \mathbf{R}_m^m$ and $B = [v \cdot w_j] \in \mathbf{R}^m$.

(c) Suppose $V = C[a, b]$ is the **inner product space** of all continuous functions $f : [a, b] \to \mathbf{R}$ with inner product $\langle f, g \rangle = \int_a^b f(t)g(t)dt$. Write the statement of the Cauchy-Schwarz Inequality for any $f, g \in V$ in terms of integrals.

Solution: The Cauchy-Schwarz Inequality in general says $|\langle u, v \rangle| \le ||u||\ ||v||$, so in $V = C[a, b]$ with $\langle f, g \rangle = \int_a^b f(t)g(t)dt$ it says $|\int_a^b f(t)g(t)dt| \le \sqrt{\int_a^b f(t)^2 dt}\ \sqrt{\int_a^b g(t)^2 dt}$.

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(3) (20 Pts) The set $S = \{1, t, t^2\}$ is the standard basis for the inner product space $\mathbf{P}_2[t]$ with inner product $\langle p, q\rangle = \int_0^1 p(t)q(t)dt$. Use the Gram-Schmidt process to convert S into an **orthogonal** basis for $\mathbf{P}_2[t]$, but **do not** bother to do the normalizing step.

Solution: For $i, j \in \{0, 1, 2\}$ we have

$$\langle t^i, t^j\rangle = \int_0^1 t^{i+j}dt = \frac{t^{i+j+1}}{i+j+1}\Big|_0^1 = \frac{1}{i+j+1}$$

so the matrix

$$M = [\langle t^i, t^j\rangle] = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

The first step of Gram-Schmidt orthogonalization does not change the first standard basis vector 1, but the second step does change t to

$$t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle}1 = t - \frac{\frac{1}{2}}{1}1 = t - \frac{1}{2}.$$

The third step changes t^2 to

$$t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}1 - \frac{\langle t^2, t - \frac{1}{2}\rangle}{\langle t - \frac{1}{2}, t - \frac{1}{2}\rangle}(t - \frac{1}{2}) = t^2 - \frac{1}{3} - \frac{\frac{1}{4} - \frac{1}{2}\frac{1}{3}}{\frac{1}{3} - \frac{1}{2}\frac{1}{4}}(t - \frac{1}{2}) =$$
$$t^2 - \frac{1}{3} - \frac{\frac{1}{12}}{\frac{1}{12}}(t - \frac{1}{2}) = t^2 - \frac{1}{3} - (t - \frac{1}{2}) = t^2 - t + \frac{1}{6}.$$

Then the orthogonal basis of V we obtained by Gram-Schmidt from S is

$$\{1, t - \frac{1}{2}, t^2 - t + \frac{1}{6}\}.$$

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(4) (15 Pts) Suppose $A, B \in \mathbf{R}_n^n$ are **orthogonal** matrices.

(a) Show that AB is also orthogonal.

Solution: Given that $A^T = A^{-1}$ and $B^T = B^{-1}$ we have
$(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}$ so AB is also orthogonal.

(b) Show that A^{-1} is also orthogonal.

Solution: A^{-1} is orthogonal since $(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$.

(c) Show that any power A^k is also orthogonal for any integer k.

Solution: In part (a) using $B = A$ we get A^2 is orthogonal. Using $B = A^2$ we get A^3 is orthogonal. Using $B = A^{k-1}$, we get A^k is orthogonal for any $k \geq 1$. Part (b) gave us A^{-1} is orthogonal, so for any positive k we have $A^{-k} = (A^{-1})^k$ is orthogonal.

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(5) (20 Pts) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) (8 Pts) The characteristic polynomial is $\Delta_A(t) = \det(\lambda I_4 - A) = (-1)^4 \det(A - \lambda I_4) =$

$$\det \begin{bmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = \det \begin{bmatrix} -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & \lambda & 0 \\ 1 & 1 & 1-\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}$$

$$= \lambda^3 \det \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 1-\lambda & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \lambda^3 \det \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \lambda^3 \, (\lambda - 3)^1 = \lambda^4 - 3\lambda^3.$$

So the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$, the roots of $\Delta_A(\lambda)$, with algebraic multiplicities $k_1 = 3$ and $k_2 = 1$.

(b) (4 Pts) For $\lambda_1 = 0$, the eigenspace, A_{λ_1}, is found by row reducing $[A - 0I_4|0]$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = -r - s - t \\ x_2 = r \in \mathbf{R} \\ x_3 = s \in \mathbf{R} \\ x_2 = t \in \mathbf{R} \end{array}$$

so the $\lambda_1 = 0$ eigenspace

$$A_{\lambda_1} = \left\{ \begin{bmatrix} -r-s-t \\ r \\ s \\ t \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r, s, t \in \mathbf{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and $g_1 = 3 = k_1$.

(4 Pts) For $\lambda_2 = 3$, the eigenspace, A_{λ_2}, is found by row reducing $[A - 3I_4|0]$:

$$\left[\begin{array}{cccc|c} -2 & 1 & 1 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = r \\ x_2 = r \\ x_3 = r \in \mathbf{R} \\ x_4 = 0 \end{array}$$

so the $\lambda_1 = 3$ eigenspace

$$A_{\lambda_2} = \left\{ \begin{bmatrix} r \\ r \\ r \\ 0 \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r \in \mathbf{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } g_2 = 1 = k_2.$$

(c) (4 Pts) A is diagonalizable since $g_1 + g_2 = 4$ and we found an eigenbasis for $\mathbf{R}^4$,

$$T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}. \; D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ and}$$

$$P = {}_S P_T = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ is the transition matrix such that $D = P^{-1}AP$ is diagonal. As a numerical check:

$$PD = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = AP.$$