Math 330-3 Number Systems Fall 2022 Exam 2 Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS. $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$

(1) (20 Points) Let $m \in \mathbb{N}^+$ and let $a, b \in \mathbb{Z}$ be such that $gcd(a, m) = 1$ and $gcd(b, m) = 1$. Prove that $gcd(ab, m) = 1$.
(HINT: If $gcd(ab, m) = c > 1$ then there is a prime number p such that $p|c$.)

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(2) (20 Points) Write each of the following as a **rational number** or a **rational number to a power**. Don't bother to reduce the rational number.

$$(a)\ \sum_{i=0}^{50} \binom{50}{i} (0.9)^i \qquad (b)\ \sum_{i=0}^{100} \binom{100}{i} \left(\frac{3}{4}\right)^i \qquad (c)\ 12.\overline{3412} \qquad (d)\ 0.0\overline{91}$$

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(3) (20 Points) Rigorously prove each of the following statements. You may use Euler's Lemma, which says that if p is prime and $a, b \in \mathbb{Z}$ such that $p|(ab)$ then $p|a$ or $p|b$.

$$(a)\ \sqrt{10} \notin \mathbb{Q} \qquad (b)\ \sqrt[3]{5} \notin \mathbb{Q}$$

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(4) (20 Points) For $m \in \mathbb{Z}$ let $m\mathbb{Z} = \{ma \mid a \in \mathbb{Z}\}$. Determine whether each set equality below is true or false. If it is true, prove it. If it is false, show why.

(a) $(6\mathbb{Z}) \cap (10\mathbb{Z}) = 60\mathbb{Z}$. (b) $(30\mathbb{Z}) \cap (70\mathbb{Z}) = 210\mathbb{Z}$.

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(5) (20 Points) Use the recursive definition $\binom{0}{0} = 1$ and $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$ to answer the following questions.

(a) Find a and b such that $\binom{m}{k-1} + 2\binom{m}{k} + \binom{m}{k+1} = \binom{a}{b}$.

(b) Prove by induction that for $2 \le m \in \mathbb{N}$, we have $\binom{m}{2} = \dfrac{m(m-1)}{2}$.

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(1) (20 Points) Let $m \in \mathbb{N}^+$ and let $a, b \in \mathbb{Z}$ be such that $gcd(a, m) = 1$ and $gcd(b, m) = 1$. Prove that $gcd(ab, m) = 1$.
(HINT: If $gcd(ab, m) = c > 1$ then there is a prime number p such that $p|c$.)

Solution: If $gcd(ab, m) = c > 1$ then there is a prime number p such that $p|c$. Since c divides both ab and m, and $p|c$, we have $p|(ab)$ and $p|m$. By Euler's lemma, $p|a$ or $p|b$, so either $p|a$ and $p|m$, or $p|b$ and $p|m$. In the first case, $p|gcd(a, m)$ says $p|1$, which is impossible, and in the second case, $p|gcd(b, m)$ says $p|1$, which is impossible. Since both cases lead to a contradiction, we must have $c = 1$.

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(2) (20 Points) Write each of the following as a **rational number** or a **rational number to a power**. Don't bother to reduce the rational number.

(a) $\sum_{i=0}^{50} \binom{50}{i} (0.9)^i = (1.9)^{50} = \left(\frac{19}{10}\right)^{50}$ from the binomial expansion of $(1 + 0.9)^{50}$.

(b) $\sum_{i=0}^{100} \binom{100}{i} \left(\frac{3}{4}\right)^i = \left(\frac{7}{4}\right)^{100} = (1.75)^{100}$ from the binomial expansion of $(1 + 0.75)^{100}$.

(c) If $x = 12.\overline{3412}$ then $10000x = 123412.\overline{3412}$ so $9999x = 123412 - 12 = 123400$ and so $x = \frac{123400}{9999}$.

(d) If $x = 0.0\overline{91}$ then $100x = 9.1\overline{91}$ so $99x = 9.1 - 0.0 = 9.1$ and so $x = \frac{9.1}{99} = \frac{91}{990}$.

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(3) (20 Points) Rigorously prove each of the following statements. You may use Euler's Lemma, which says that if p is prime and $a, b \in \mathbb{Z}$ such that $p|(ab)$ then $p|a$ or $p|b$.

(a) If $\sqrt{10} \in \mathbb{Q}$ then $\sqrt{10} = \frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $gcd(m, n) = 1$. Then we would have $10 = \frac{m^2}{n^2}$ so $10n^2 = m^2$. Since 2 is prime and $2|10$, we have $2|m^2$, so Euler's Lemma says $2|m$ so $m = 2a$ for some $a \in \mathbb{Z}$ and then $10n^2 = m^2 = 4a^2$ gives $5n^2 = 2a^2$. This says $2|(5n^2)$, but 2 does not divide 5, so by Euler's Lemma, $2|n^2$ and so $2|n$. But $2|m$ and $2|m$ contradicts $gcd(m, n) = 1$. This proves by contradiction that $\sqrt{10} \notin \mathbb{Q}$.

(b) If $\sqrt[3]{5} \in \mathbb{Q}$ then $\sqrt[3]{5} = \frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $gcd(m, n) = 1$. Then we would have $5 = \frac{m^3}{n^3}$ so $5n^3 = m^3$. Since 5 is prime and $5|m^3$, Euler's Lemma says $5|m$ so $m = 5a$ for some $a \in \mathbb{Z}$ and then $5n^3 = m^3 = 125a^3$ gives $n^3 = 25a^3$. Since $5|25$ this says $5|n^3$, so by Euler's Lemma we get $5|n$. But $5|m$ and $5|n$ contradicts $gcd(m, n) = 1$. This proves by contradiction that $\sqrt[3]{5} \notin \mathbb{Q}$.

(4) (20 Points) For $m \in \mathbb{Z}$ let $m\mathbb{Z} = \{ma \mid a \in \mathbb{Z}\}$. Determine whether each set equality below is true or false. If it is true, prove it. If it is false, show why.

(a) $(6\mathbb{Z}) \cap (10\mathbb{Z}) = 60\mathbb{Z}$. (b) $(30\mathbb{Z}) \cap (70\mathbb{Z}) = 210\mathbb{Z}$.

(a) False. Although $60\mathbb{Z} \subset (6\mathbb{Z}) \cap (10\mathbb{Z})$ because $60|a$ implies $6|a$ and $10|a$, they are not equal because $30 \in (6\mathbb{Z}) \cap (10\mathbb{Z})$ but $30 \notin 60\mathbb{Z}$. In fact, $(6\mathbb{Z}) \cap (10\mathbb{Z}) = 30\mathbb{Z}$.

(b) True. Let $a \in (30\mathbb{Z}) \cap (70\mathbb{Z})$, so $30|a$ and $70|a$, that is, $a = 30m = 70n$ for some $m, n \in \mathbb{Z}$. Writing these in terms of prime factors, $a = (2)(3)(5)m = (2)(5)(7)n$. This implies $3|n$ so $n = 3r$ for some $r \in \mathbb{Z}$ so $a = (2)(5)(7)(3r) \in 210\mathbb{Z}$ giving $(30\mathbb{Z}) \cap (70\mathbb{Z}) \subseteq 210\mathbb{Z}$. Let $b \in 210\mathbb{Z}$ so $210|b$. Since $30|210$ and $70|210$, we get $30|b$ and $70|b$ so $b \in (30\mathbb{Z}) \cap (70\mathbb{Z})$, giving the reverse containment, and therefore, equality.

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(5) (20 Points) Use the recursive definition $\binom{0}{0} = 1$ and $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$ to answer the following questions.

(a) We have $\binom{m}{k-1} + 2\binom{m}{k} + \binom{m}{k+1} = \binom{m}{k-1} + \binom{m}{k} + \binom{m}{k} + \binom{m}{k+1}$ and using the recursive definition of "choose", the sum of the first two terms is $\binom{m}{k-1} + \binom{m}{k} = \binom{m+1}{k}$ while the sum of the second two terms is $\binom{m}{k} + \binom{m}{k+1} = \binom{m+1}{k+1}$. So adding those, we get

$$\binom{m+1}{k} + \binom{m+1}{k+1} = \binom{m+2}{k+1} = \binom{a}{b}.$$

(b) For $2 \le m \in \mathbb{N}$, let $P(m)$ be the assertion that $\binom{m}{2} = \frac{m(m-1)}{2}$.

The base case $P(2)$ says $\binom{2}{2} = \frac{2(2-1)}{2}$ which is true since $\binom{2}{2} = 1 = \frac{2(1)}{2}$.

Assume $P(m)$ is true for some $2 \le m \in \mathbb{N}$ and try to prove $P(m+1)$, which says $\binom{m+1}{2} = \frac{(m+1)(m)}{2}$. By the recursive definition and $P(m)$, we have

$$\binom{m+1}{2} = \binom{m}{2} + \binom{m}{1} = \frac{m(m-1)}{2} + m = \frac{m(m-1)+2m}{2} = \frac{m^2+m}{2} = \frac{(m+1)(m)}{2}.$$

We have shown that $P(m)$ implies $P(m+1)$, so the assertion is proved by induction for all $2 \le m \in \mathbb{N}$.