Math 330-3 Number Systems Fall 2022 Exam 3 Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS.

$\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$ and $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r > 0\}$

Def: For any sequence of rational numbers a_n, $n \in \mathbb{N}^+$, we say $\lim_{n\to\infty} a_n = L$ when $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $n \geq M_\epsilon$ then $|a_n - L| < \epsilon$.

Def: We say a sequence of rational numbers a_n, $n \in \mathbb{N}^+$, is **Cauchy** when $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $m, n \geq M_\epsilon$ then $|a_m - a_n| < \epsilon$.

Archimedean Lemma: $\forall x \in \mathbb{R}$, $\exists N_x \in \mathbb{N}^+$ such that $x < N_x$.

Triangle Inequality For any $a, b \in \mathbb{Q}$ we have $|a + b| \leq |a| + |b|$.

Def: For $n \in \mathbb{N}^+$ define $[1, n] = \{i \in \mathbb{N} \mid 1 \leq i \leq n\} = \{1, \cdots, n\}$.

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(1) (20 Points) Answer the following questions about rational sequences.

(a) If $\lim_{n\to\infty} a_n = L$ then the sequence a_n is Cauchy.
Hint: For any $m, n \in \mathbb{N}^+$, $|a_m - a_n| = |(a_m - L) - (a_n - L)|$.

(b) Use the **definition** of limit to prove that $a_n = \dfrac{3n+1}{4n+1}$ has $\lim_{n\to\infty} a_n = \dfrac{3}{4}$.

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(2) (20 Points) Suppose a jar contains 15 marbles, each marked with a different number from 1 to 15.

(a) How many different ways are there to arrange 5 marbles from the full jar into a line?

(b) How many different subsets of 12 marbles can be made from a full jar?

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(3) (20 Points) We say two sets, S and T, have the same **cardinality** when there exists some bijection $f : S \to T$.

(a) Prove that the set of all even integers, $2\mathbb{Z}$, has the same cardinality as the set of all odd integers, $2\mathbb{Z} + 1$.

(b) For $m > n$ in $\mathbb{N}^+$, prove that $[1, m]$ and $[1, n]$ do not have the same cardinality.

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(4) (20 Points) Write each of the following expressions as a rational number or a power of a rational number.

$(a)\ 1.\overline{126}$ $\qquad (b)\ 0.\overline{10}$ $\qquad (c)\ \sum_{i=0}^{200} \binom{200}{i}$ $\qquad (d)\ \sum_{i=0}^{50} \binom{50}{i} \left(\frac{3}{5}\right)^i$

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(5) (20 Points) Let S be a non-empty subset of $\mathbb{Q}_{\geq 0} = \{x \in \mathbb{Q} \mid x \geq 0\}$. We say S has a **least element** when $\exists s_{min} \in S$ such that $\forall s \in S$, $s_{min} \leq s$. We say S has an **upper bound** in $\mathbb{Q}$ when $\exists U \in \mathbb{Q}$ such that $\forall s \in S$, $s \leq U$. For each statement below, **give a counter-example to show it is false**.

(a) Any non-empty subset S of $\mathbb{Q}_{\geq 0}$ has a least element.

(b) Let T be a non-empty subset of $\mathbb{Q}_{\geq 0}$ with an upper bound $U \in \mathbb{Q}$. Let $UB(T)$ be the set of all rational upper bounds of T. Then $UB(T)$ has a least element, that is T has a **least upper bound** in $\mathbb{Q}$.

Def: For any sequence of rational numbers a_n, $n \in \mathbb{N}^+$, we say $\lim_{n\to\infty} a_n = L$ when $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $n \geq M_\epsilon$ then $|a_n - L| < \epsilon$.

Def: We say a sequence of rational numbers a_n, $n \in \mathbb{N}^+$, is **Cauchy** when $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $m, n \geq M_\epsilon$ then $|a_m - a_n| < \epsilon$.

Archimedean Lemma: $\forall x \in \mathbb{R}$, $\exists N_x \in \mathbb{N}^+$ such that $x < N_x$.

Triangle Inequality For any $a, b \in \mathbb{Q}$ we have $|a + b| \leq |a| + |b|$.

Def: For $n \in \mathbb{N}^+$ define $[1, n] = \{i \in \mathbb{N} \mid 1 \leq i \leq n\} = \{1, \cdots, n\}$.

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(1) (20 Points) Answer the following questions about rational sequences.

(a) If $\lim_{n\to\infty} a_n = L$ then the sequence a_n is Cauchy.
Hint: For any $m, n \in \mathbb{N}^+$, $|a_m - a_n| = |(a_m - L) - (a_n - L)|$.

Solution: We know that $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $n \geq M_\epsilon$ then $|a_n - L| < \epsilon$. So using the hint and the Triangle Inequality, for any $\epsilon \in \mathbb{Q}^+$, $\exists M_{\epsilon/2} \in \mathbb{N}^+$ such that if $m, n \geq M_{\epsilon/2}$ then $|a_m - L| < \epsilon/2$ and $|a_n - L| < \epsilon/2$ so
$|a_m - a_n| = |(a_m - L) - (a_n - L)| \leq |a_m - L| + |a_n - L| < \epsilon/2 + \epsilon/2 = \epsilon$.

(b) Use the **definition** of limit to prove that $a_n = \dfrac{3n+1}{4n+1}$ has $\lim_{n\to\infty} a_n = \dfrac{3}{4}$.

Solution: The definition of limit requires us to prove that
$\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $n \geq M_\epsilon$ then $\left|a_n - \frac{3}{4}\right| < \epsilon$. We have

$$\left|a_n - \frac{3}{4}\right| = \left|\frac{3n+1}{4n+1} - \frac{3}{4}\right| = \left|\frac{4(3n+1) - 3(4n+1)}{(4n+1)4}\right| = \left|\frac{1}{4(4n+1)}\right| = \frac{1}{16n+4} < \frac{1}{16n}.$$

Since $n \geq M_\epsilon$ iff $\frac{1}{n} \leq \frac{1}{M_\epsilon}$ iff $\frac{1}{16n} \leq \frac{1}{16M_\epsilon}$, we want to find M_ϵ such that $\frac{1}{16M_\epsilon} < \epsilon$. But that is equivalent to $\frac{1}{16\epsilon} < M_\epsilon$. We can always find such a natural number by the Archimedean Lemma, completing the proof.

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(2) (20 Points) Suppose a jar contains 15 marbles, each marked with a different number from 1 to 15.

(a) How many different ways are there to arrange 5 marbles from the full jar into a line?

Solution: There are 15 choices for the first marble in line, 14 remaining choices for the second marble, 13 remaining choices for the third marble, 12 choices for the fourth marble, and 11 choices for the fifth marble. So the total number of ways to arrange 5 marbles chosen from the full jar is $(15)(14)(13)(12)(11) = 360360$.

(b) How many different subsets of 12 marbles can be made from a full jar?

Solution: The number of subsets of size 12 taken from a set of 15 objects is given by the binomial coefficient $\binom{15}{12} = \dfrac{15!}{(12!)(3!)} = \dfrac{(15)(14)(13)}{6} = 455$.

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(3) (20 Points) We say two sets, S and T, have the same **cardinality** when there exists some bijection $f : S \to T$.

(a) Prove that the set of all even integers, $2\mathbb{Z}$, has the same cardinality as the set of all odd integers, $2\mathbb{Z}+1$.

Solution: The function $f : 2\mathbb{Z} \to 2\mathbb{Z}+1$ defined by $f(2n) = 2n+1$ for any $n \in \mathbb{Z}$ is injective since $f(2m) = f(2n)$ means $2m+1 = 2n+1$ so $2m = 2n$ so $m = n$. It is also surjective since by definition $2\mathbb{Z}+1 = \{2n+1 \mid n \in \mathbb{Z}\} = Range(f)$, so f is bijective.

(b) For $m > n$ in $\mathbb{N}^+$, prove that $[1, m]$ and $[1, n]$ do not have the same cardinality.

Solution: By contradiction, suppose there were a bijection $f : [1, m] \to [1, n]$. Since f is injective, the m distinct elements of $[1, m]$ are sent to m distinct elements, $f(1), \cdots, f(m)$ in $[1, n]$. But since $m > n$, there are more elements in $Range(f)$ than there are in $[1, n]$.

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(4) (20 Points) Write each of the following expressions as a rational number or a power of a rational number.

(a) If $x = 1.\overline{126}$ then $1000x = 1126.\overline{126}$ so $999x = 1125$ so $x = \frac{1125}{999} = \frac{125}{111}$.

(b) If $x = 0.\overline{10}$ then $100x = 10.\overline{10}$ so $99x = 10$ so $x = \frac{10}{99}$.

(c) From the binomial expansion, $2^{200} = (1+1)^{200} = \sum_{i=0}^{200} \binom{200}{i} 1^{200-i} 1^i = \sum_{i=0}^{200} \binom{200}{i}$.

(d) Using $x = 5$, $y = 3$ and $m = 50$ in the binomial expansion, we have

$$(5+3)^{50} = \sum_{i=0}^{50} \binom{50}{i} 5^{50-i} 3^i = 5^{50} \sum_{i=0}^{50} \binom{50}{i} 5^{-i} 3^i = 5^{50} \sum_{i=0}^{50} \binom{50}{i} \left(\frac{3}{5}\right)^i.$$

Dividing by 5^{50} we get $\left(\frac{8}{5}\right)^{50} = \sum_{i=0}^{50} \binom{50}{i} \left(\frac{3}{5}\right)^i$.

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(5) (20 Points) Let S be a non-empty subset of $\mathbb{Q}_{\geq 0} = \{x \in \mathbb{Q} \mid x \geq 0\}$. We say S has a **least element** when $\exists s_{min} \in S$ such that $\forall s \in S$, $s_{min} \leq s$. We say S has an **upper bound** in $\mathbb{Q}$ when $\exists U \in \mathbb{Q}$ such that $\forall s \in S$, $s \leq U$. For each statement below, **give a counter-example to show it is false**.

(a) Any non-empty subset S of $\mathbb{Q}_{\geq 0}$ has a least element.

Solution: $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ is a non-empty subset of $\mathbb{Q}_{\geq 0}$ which has no least element. No $0 < r \in \mathbb{Q}^+$ can be the smallest since $0 < \frac{r}{2} \in \mathbb{Q}^+$ is smaller.

(b) Let T be a non-empty subset of $\mathbb{Q}_{\geq 0}$ with an upper bound $U \in \mathbb{Q}$. Let $UB(T)$ be the set of all rational upper bounds of T. Then $UB(T)$ has a least element, that is T has a **least upper bound** in $\mathbb{Q}$.

Solution: Let $T = \{x \in \mathbb{Q}^+ \mid x^2 < 2\}$. Then $UB(T) = \{U \in \mathbb{Q}^+ \mid 2 \leq U^2\} = \{U \in \mathbb{Q}^+ \mid \sqrt{2} \leq U\}$. There are $U \in \mathbb{Q}^+$ above $\sqrt{2}$ as close as you like, but no least one.