Math 330-3 Number Systems Fall 2022 Practice Exam 1 Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS. $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$

1. (20 Points) Write each of the following statements P as a logically equivalent statement Q so that the word "not" does not occur anywhere in Q. Whether the statement is true or false is not the issue.

   (a) not($\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $x < y$)
   (b) not($\exists x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $xy = 12$)
   (c) not($x \leq y$ or $10 < x + y$)
   (d) not(if $x > y$ then $x^2 > y^2$)

2. (20 Points) We say $a \in \mathbb{Z}$ is even when $2|a$, and we say it is odd when a is not even. For $m, n \in \mathbb{Z}$ we define $n > m$ when $n - m \in \mathbb{N}^+$ and $n \geq m$ when $n - m \in \mathbb{N}$.

   (a) Use that for $m, n \in \mathbb{N}$, if $m|n$ then $m \leq n$ to prove that 1 is odd.
   (b) Prove that for any $n \in \mathbb{Z}$, $2n + 1$ is odd.

3. (20 Points) Determine whether each assertion is true or false. If it is true, prove it. If it is false, give a counterexample.

   (a) $\forall m \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$ such that $m - n = 2$.
   (b) $\exists n \in \mathbb{Z}$ such that $\forall m \in \mathbb{Z}$ we have $m - n = 2$.

4. (20 Points) Prove each of the following statements rigorously. You may use that $b^m b^k = b^{m+k}$ and $(b^m)^k = b^{mk}$ for any $b \in \mathbb{Z}$, $m, k \in \mathbb{N}$.

   (a) For any $n \in \mathbb{N}$, $1^n = 1$.
   (b) For any $n \in \mathbb{N}$, $(-1)^n = 1$ if n is even, $(-1)^n = -1$ if n is odd.
   (c) For any $a, b \in \mathbb{Z}$ and any $m \in \mathbb{N}$, $(a^m)(b^m) = (ab)^m$.
   Explicitly show in your proof each step where you used associativity or commutativity of multiplication in $\mathbb{Z}$.

5. (20 Points) Prove by induction that for any $n \in \mathbb{N}$, $2n^3 + 3n^2 + n$ is divisible by 6.

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1. (20 Points) Write each of the following statements P as a logically equivalent statement Q so that the word "not" does not occur anywhere in Q. Whether the statement is true or false is not the issue.

   (a) not($\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $x < y$) is equivalent to $\exists x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}$, $x \geq y$.
   (b) not($\exists x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $xy = 12$) is equivalent to $\forall x \in \mathbb{Z}$, $\forall y \in \mathbb{Z}$, $xy \neq 12$.
   (c) not($x \leq y$ or $10 < x + y$) is equivalent to $x > y$ and $10 \geq x + y$.
   (d) not(if $x > y$ then $x^2 > y^2$) is equivalent to $x > y$ and $x^2 \leq y^2$.

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2. (20 Points) We say $a \in \mathbb{Z}$ is even when $2|a$, and we say it is odd when a is not even. For $m, n \in \mathbb{Z}$ we define $n > m$ when $n - m \in \mathbb{N}^+$ and $n \geq m$ when $n - m \in \mathbb{N}$.

   (a) Use that for $m, n \in \mathbb{N}$, if $m|n$ then $m \leq n$ to prove that 1 is odd.

Proof: Suppose that 1 were even, so $2|1$, so $2 \leq 1$. But $2 > 1$ since $2 - 1 = 1 \in \mathbb{N}$. This contradiction shows that 1 is odd.

   (b) Prove that for any $n \in \mathbb{Z}$, $2n + 1$ is odd.

Proof: Suppose to the contrary that for some $n \in \mathbb{Z}$, $2n + 1$ is even, so there exists an $a \in \mathbb{Z}$ such that $2n + 1 = 2a$. But then $1 = 2a - 2n = 2(a - n)$ would mean 1 is even, contradicting the result of 2(a).

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3. (20 Points) Determine whether each assertion is true or false. Prove your answer.

   (a) $\forall m \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$ such that $m - n = 2$.

True. Proof: For any $m \in \mathbb{Z}$, we see that $n = m - 2 \in \mathbb{Z}$ satisfies $m - n = m - (m - 2) = m + (-m) - (-2) = 0 - (-2) = 2$.

   (b) $\exists n \in \mathbb{Z}$ such that $\forall m \in \mathbb{Z}$ we have $m - n = 2$.

False. Proof: By contradiction, suppose such an integer n exists, then in particular for $m = n$ we would have $0 = n - n = 2$, a contradiction.

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4. (20 Points) Prove each of the following statements rigorously. You may use that $b^m b^k = b^{m+k}$ and $(b^m)^k = b^{mk}$ for any $b \in \mathbb{Z}$, $m, k \in \mathbb{N}$.

   (a) For any $n \in \mathbb{N}$, $1^n = 1$.

Proof: For $n \in \mathbb{N}$, let $P(n)$ be the statement $1^n = 1$. $P(0)$ is true since by definition $1^0 = 1$. Assume $P(n)$. Using the definition of powers, and then using $P(n)$ we have $1^{n+1} = (1^n)(1) = (1)(1) = 1$, so $P(n + 1)$ is true. Done by induction.

   (b) For any $n \in \mathbb{N}$, $(-1)^n = 1$ if n is even, $(-1)^n = -1$ if n is odd.

Proof: If $n \in \mathbb{N}$ is even then $n = 2k$ for some $k \in \mathbb{N}$, so $(-1)^n = (-1)^{2k} = ((-1)^2)^k$. But $(-1)^2 = (-1)(-1) = 1$ so $(-1)^n = 1^k = 1$ by 4(a). If $n \in \mathbb{N}$ is odd then $n = 2k + 1$ for some $k \in \mathbb{N}$, so $(-1)^n = (-1)^{2k+1} = (-1)^{2k}(-1)^1 = (1)(-1) = -1$.

(c) For any $a, b \in \mathbb{Z}$ and any $m \in \mathbb{N}$, $(a^m)(b^m) = (ab)^m$.
Explicitly show in your proof each step where you used associativity or commutativity of multiplication in $\mathbb{Z}$.

Proof: For $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$ let $P(m)$ be the statement $(a^m)(b^m) = (ab)^m$. $P(0)$ is true since $(a^0)(b^0) = (1)(1) = 1 = (ab)^0$. Assume $P(m)$. Then $(a^{m+1})(b^{m+1}) = (a^m a)(b^m b)$ by definition of powers. Using associativity and commutativity of multiplication in $\mathbb{Z}$, we have $(a^m a)(b^m b) = a^m(a(b^m b)) = a^m((b^m b)a) = a^m(b^m(ba)) = (a^m b^m)(ba) = (a^m b^m)(ab)$. Using $P(m)$, and the definition of powers, we have $(a^m b^m)(ab) = (ab)^m(ab) = (ab)^{m+1}$. This gives $P(m+1)$, so we are done by induction.

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5. (20 Points) Prove by induction that for any $n \in \mathbb{N}$, $2n^3 + 3n^2 + n$ is divisible by 6.

Proof: For $n \in \mathbb{N}$ let $P(n)$ be the statement $6|(2n^3 + 3n^2 + n)$. $P(0)$ is true since $2(0^3) + 3(0^2) + 0 = 0 + 0 + 0 = 0$ and $6|0$ is true. Assume $P(n)$ is true, so for some $a \in \mathbb{Z}$, $2n^3 + 3n^2 + n = 6a$. Show that $P(n+1)$ is true. Using algebraic properties of the integers, we have

$$
\begin{aligned}
2(n+1)^3 + 3(n+1)^2 + (n+1) &= 2(n^3 + 3n^2 + 3n + 1) + 3(n^2 + 2n + 1) + (n+1) \\
&= (2n^3 + 3n^2 + n) + 2(3n^2 + 3n + 1) + 3(2n + 1) + 1 \\
&= 6a + 6n^2 + 6n + 2 + 6n + 3 + 1 \\
&= 6a + 6(n^2 + 2n + 1) \\
&= 6(a + n^2 + 2n + 1)
\end{aligned}
$$

is divisible by 6. So $P(n+1)$ is true. We are done by induction.