Math 330-3 Number Systems Fall 2022 Practice Problems for Exam 2 Feingold

No solution guide is being provided for these problems, but answers to some of them can be found in our textbook. After you have worked on these, if you still have questions, they can be discussed in class before the exam is given.

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(1) For all $m \in \mathbb{N}$, explain why

$$\sum_{k=0}^{m} \binom{m}{k} 4^k = \sum_{k=0}^{m} \binom{m}{k} 3^k 2^{m-k}$$

and express their common value as a single integer (not involving any sums).

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(2) Prove that for any $3 \leq m \in \mathbb{N}$, we have $\binom{m}{3} = \frac{m(m-1)(m-2)}{3!}$

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(3) For $m \in \mathbb{Z}$ let $m\mathbb{Z} = \{ma \mid a \in \mathbb{Z}\}$. Determine whether each set equality below is true or false. If it is true, prove it. If it is false, show why.

(a) $(3\mathbb{Z}) \cap (5\mathbb{Z}) = 15\mathbb{Z}$ (b) $(12\mathbb{Z}) \cap (15\mathbb{Z}) = 180\mathbb{Z}$ (c) $(12\mathbb{Z}) \cap (15\mathbb{Z}) = 60\mathbb{Z}$.

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(4) A function $F : A \to B$ is said to be **injective** when $\forall a_1, a_2 \in A$, if $F(a_1) = F(a_2)$ then $a_1 = a_2$. F is said to be **surjective** when $\forall b \in B$, $\exists a \in A$ such that $F(a) = b$. For each of the functions below, determine whether or not the function is injective, whether or not the function is surjective, and justify your answers rigorously.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 5n - 7$ (b) $g : \mathbb{Z} \to \mathbb{Z}$ by $g(m) = m^2$

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(5) Prove that $\sqrt{3} \notin \mathbb{Q}$.

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(6) Let $m \in \mathbb{N}^+$ and let $a \in \mathbb{Z}$ be such that $gcd(a, m) = 1$. Prove that there exists some $x \in \mathbb{Z}$ such that $ax \equiv 1 \pmod{m}$ and $gcd(x, m) = 1$.
What does this mean about $U(m) = \{[a]_m \in \mathbb{Z}_m \mid gcd(a, m) = 1\}$?
What must be checked to prove that $U(m)$ is closed under multiplication mod m?

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(7) Use the Fundamental Theorem of Arithmetic to show that there must be an infinite number of primes. **Hint**: Suppose there are only a finite number of primes, $p_1, p_2, \cdots, p_r$. What can you say about the number $(p_1 p_2 \cdots p_r) + 1$?

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(8) Use the binomial expansion to express the sum $\sum_{i=0}^{5} \sum_{j=0}^{8} \binom{5}{i} \binom{8}{j} 2^{2i+8-j}\, 3^{5-i+j}$ in the form $m^r n^s$ for some $m, n, r, s \in \mathbb{N}$. **Hint**: Write this as the product of two sums.
