NAME (Printed): ______________________________

Math 330-3 Number Systems Fall 2022 Quiz 2 Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS. $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$

Define the "Fibonacci" sequence recursively by:

$$F_0 = 0, \quad F_1 = 1 \text{ and } \forall k \in \mathbb{N}, \ F_{k+2} = F_{k+1} + F_k.$$

1. (5 points) Use that definition to compute the first 10 values of the sequence, and fill in the following table with your answers:

k	0	1	2	3	4	5	6	7	8	9
F_k	0	1								

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2. (5 points) Compute F_{k+1}^2 and $F_k F_{k+2}$ for $0 \le k \le 7$ and fill in the following table with your answers:

k	0	1	2	3	4	5	6	7
F_{k+1}^2								
$F_k F_{k+2}$								

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3. (10 points) This data should support the conjecture that $F_{k+1}^2 - F_k F_{k+2} = (-1)^k$ for $k \in \mathbb{N}$. Prove this by induction on k. **Hint:** For $k \in \mathbb{N}$ let $P(k)$ be the above formula. In the inductive step, assuming $P(k)$ is true for some $k \in \mathbb{N}$, try to prove $P(k+1)$, that is, prove $F_{k+2}^2 - F_{k+1}F_{k+3} = (-1)^{k+1}$. Starting with the left hand side of that equation, use the recursive definition to replace F_{k+3} by $F_{k+2} + F_{k+1}$ and then use algebra and $P(k)$ to get the right side of $P(k+1)$.

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Define the "Fibonacci" sequence recursively by:

$$F_0 = 0, \quad F_1 = 1 \text{ and } \forall k \in \mathbb{N}, \quad F_{k+2} = F_{k+1} + F_k.$$

1. (5 points) Use that definition to compute the first 10 values of the sequence, and fill in the following table with your answers:

k	0	1	2	3	4	5	6	7	8	9
F_k	0	1	1	2	3	5	8	13	21	34

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2. (5 points) Compute F_{k+1}^2 and F_kF_{k+2} for $0 \leq k \leq 7$ and fill in the following table with your answers:

k	0	1	2	3	4	5	6	7
F_{k+1}^2	1	1	4	9	25	64	169	441
F_kF_{k+2}	0	2	3	10	24	65	168	442

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3. (10 points) This data should support the conjecture that $F_{k+1}^2 - F_kF_{k+2} = (-1)^k$ for $k \in \mathbb{N}$. Prove this by induction on k. **Hint:** For $k \in \mathbb{N}$ let $P(k)$ be the above formula. In the inductive step, assuming $P(k)$ is true for some $k \in \mathbb{N}$, try to prove $P(k+1)$, that is, prove $F_{k+2}^2 - F_{k+1}F_{k+3} = (-1)^{k+1}$. Starting with the left hand side of that equation, use the recursive definition to replace F_{k+3} by $F_{k+2} + F_{k+1}$ and then use algebra and $P(k)$ to get the right side of $P(k+1)$.

The data in the table above shows that $F_{k+1}^2 - F_kF_{k+2} = (-1)^k$ for $0 \leq k \leq 7$, so for $k \in \mathbb{N}$ let $P(k)$ be that assertion. The table verifies that $P(k)$ is true for $0 \leq k \leq 7$ but all we need to start an induction is the base case $P(0)$ which says $F_1^2 - F_0F_2 = (-1)^0$, that is, $1^2 - (0)(1) = (-1)^0$ which is true.

Assume that $P(k)$ is true for some $k \in \mathbb{N}$ and try to prove $P(k+1)$, that is, prove $F_{k+2}^2 - F_{k+1}F_{k+3} = (-1)^{k+1}$. The left hand side of that equation equals

$$\begin{aligned}
F_{k+2}^2 - F_{k+1}F_{k+3} &= F_{k+2}^2 - F_{k+1}(F_{k+2} + F_{k+1}) \\
&= F_{k+2}^2 - F_{k+1}F_{k+2} - F_{k+1}^2 \\
&= (F_{k+2} - F_{k+1})F_{k+2} - F_{k+1}^2 \\
&= F_kF_{k+2} - F_{k+1}^2 \\
&= -(F_{k+1}^2 - F_kF_{k+2}) \\
&= -(-1)^k \\
&= (-1)^{k+1}.
\end{aligned}$$

This shows $P(k+1)$ is true and completes the proof by induction.
