NAME (Printed): ____

Math 330-3   Number Systems   Fall 2022   Quiz 4   Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS. $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$

Write each of the following infinite repeating decimals as a **rational number**, $\frac{m}{n}$, for some $m, n \in \mathbb{Z}$. Don't bother to write the fraction in reduced form.

1. (5 points) $8.262626\cdots = 8.\overline{26}$

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2. (5 points) $0.0123123123\cdots = 0.0\overline{123}$

3. (5 points) Fill in the following table with the powers 2^k, 3^k and $4^k \pmod{13}$. Write your answers as numbers $1 \le m \le 12$, understood as the remainders mod 13. Instead of computing high powers of 2, 3 and 4, use the recursive definition of powers to get the next number from the previous one, and then reduce it mod 13 to be in the required interval. For example, $2^4 = 2^3 \cdot 2 = 8 \cdot 2 = 16 \equiv 3 \pmod{13}$ so $2^5 \equiv 3 \cdot 2 \pmod{13}$. As soon as some power gives 1, the pattern will repeat. Note that 0 should never occur in the table.

k	1	2	3	4	5	6	7	8	9	10	11	12
2^k	2	4	8	3	6							
3^k	3	9										
4^k	4	3										

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4. (5 points) From the table **find the least** $a, b, c \in \mathbb{N}^+$ such that $2^a \equiv 1 \pmod{13}$, $3^b \equiv 1 \pmod{13}$ and $4^c \equiv 1 \pmod{13}$. Use these to **find the unique integers** $1 \le x, y, z \le 12$ such that $2^{1000} \equiv x \pmod{13}$, $3^{1000} \equiv y \pmod{13}$ and $4^{1000} \equiv z \pmod{13}$. **Hint**: First write $1000 = aq_1 + r_1 = bq_2 + r_2 = cq_3 + r_3$ where $0 \le r_1 < a$, $0 \le r_2 < b$ and $0 \le r_3 < c$.

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Write each of the following infinite repeating decimals as a **rational number**, $\frac{m}{n}$, for some $m, n \in \mathbb{Z}$. Don't bother to write the fraction in reduced form.

1. (5 points) Let $x = 8.\overline{26}$ so that $100x = 826.\overline{26}$ and $99x = 100x - x = 826.\overline{26} - 8.\overline{26} = 826 - 8 = 818$. Then we find $x = \frac{818}{99}$

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2. (5 points) Let $x = 0.0\overline{123}$ so that $1000x = 12.3\overline{123}$ and $999x = 1000x - x = 12.3\overline{123} - 0.0\overline{123} = 12.3 - 0.0 = 12.3$. Then we find $x = \frac{12.3}{999} = \frac{123}{9990}$.

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3. (5 points) Fill in the following table with the powers 2^k, 3^k and $4^k \pmod{13}$. Write your answers as numbers $1 \le m \le 12$ understood as the remainders mod 13. Instead of computing high powers of 2, 3 and 4, use the recursive definition of powers to get the next number from the previous one, and then reduce it mod 13 to be in the required interval. For example, $2^4 = 2^3 \cdot 2 = 8 \cdot 2 = 16 \equiv 3 \pmod{13}$ so $2^5 \equiv 3 \cdot 2 \pmod{13}$. As soon as some power gives 1, the pattern will repeat. Note that 0 should never occur in the table.

k	1	2	3	4	5	6	7	8	9	10	11	12
2^k	2	4	8	3	6	12	11	9	5	10	7	1
3^k	3	9	1	3	9	1	3	9	1	3	9	1
4^k	4	3	12	9	10	1	4	3	12	9	10	1

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4. (5 points) From the table, the **least** $a, b, c \in \mathbb{N}^+$ such that $2^a \equiv 1 \pmod{13}$, $3^b \equiv 1 \pmod{13}$ and $4^c \equiv 1 \pmod{13}$ are $a = 12$, $b = 3$ and $c = 6$. Since

$$1000 = (12)(83) + 4 = (3)(333) + 1 = (6)(166) + 4$$

we have

$$2^{1000} = 2^{(12)(83)+4} = (2^{12})^{83}(2^4) \equiv (1)^{83}(2^4) \equiv 3 \pmod{13} \quad \text{so } x = 3,$$

$$3^{1000} = 3^{(3)(333)+1} = (3^3)^{333}(3^1) \equiv (1)^{333}(3^1) \equiv 3 \pmod{13} \quad \text{so } y = 3,$$

$$4^{1000} = 4^{(6)(166)+4} = (4^6)^{166}(4^4) \equiv (1)^{166}(4^4) \equiv 9 \pmod{13} \quad \text{so } z = 9.$$
