NAME (Printed): ____

Math 330-3   Number Systems   Fall 2022   Quiz 6   Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS.

$\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ and $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r > 0\}$

Def: For any sequence of rational numbers a_n, $n \in \mathbb{N}^+$, we say $\lim_{n \to \infty} a_n = L$ when $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $n \geq M_\epsilon$ then $|a_n - L| < \epsilon$.

Lemma: $\forall x \in \mathbb{R}$, $\exists N_x \in \mathbb{N}^+$ such that $x < N_x$.

Using the above defintion of limit and the lemma, prove the following results about limits:

1. (10 points) $\lim_{n \to \infty} \frac{1}{n^2} = 0$

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2. (10 points) $\lim_{n \to \infty} \frac{n}{n+1} = 1$

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1. (10 points) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

   Solution: $\forall\epsilon \in \mathbb{Q}^+$ we need to show that $\exists M_\epsilon \in \mathbb{N}^+$ such that $n \geq M_\epsilon$ implies $\left|\frac{1}{n^2}\right| < \epsilon$. Since $\frac{1}{n^2} > 0$ we can drop the absolute value and say that we want to know how big n has to be in order to guarantee that $\frac{1}{n^2} < \epsilon$, which is equivalent to $\frac{1}{\epsilon} < n^2$ which means $\frac{1}{\sqrt{\epsilon}} < n$. Using the lemma with $x = \frac{1}{\sqrt{\epsilon}}$, there is some $N_x \in \mathbb{N}^+$ such that $\frac{1}{\sqrt{\epsilon}} < N_x$, so if $n \geq M_\epsilon = N_x$ we have $\frac{1}{\sqrt{\epsilon}} < M_x \leq n$ as required.

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2. (10 points) $\lim_{n\to\infty} \frac{n}{n+1} = 1$

   Solution: $\forall\epsilon \in \mathbb{Q}^+$ we need to show that $\exists M_\epsilon \in \mathbb{N}^+$ such that $n \geq M_\epsilon$ implies $\left|\frac{n}{n+1} - 1\right| < \epsilon$. We have

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$$

   so we want to know how big n has to be in order to guarantee that $\frac{1}{n+1} < \epsilon$, which means $\frac{1}{\epsilon} < n+1$ which is equivalent to $\frac{1}{\epsilon} - 1 < n$. Using the lemma with $x = \frac{1}{\epsilon} - 1$, there is some $N_x \in \mathbb{N}^+$ such that $\frac{1}{\epsilon} - 1 < N_x$ so if $n \geq M_\epsilon = N_x$ we have $\frac{1}{\epsilon} - 1 < M_x \leq n$ as required.
