NAME (Printed): ______________________________________

Math 330-3 Number Systems Fall 2022 Quiz 7 Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS.

$\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$ and $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r > 0\}$

Def: We say a sequence of rational numbers a_n, $n \in \mathbb{N}^+$, is **Cauchy** when $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $m, n \geq M_\epsilon$ then $|a_m - a_n| < \epsilon$.

Lemma: $\forall x \in \mathbb{R}$, $\exists N_x \in \mathbb{N}^+$ such that $x < N_x$.

Theorem (Triangle Inequality in $\mathbb{Q}$) For any $a, b \in \mathbb{Q}$ we have $|a + b| \leq |a| + |b|$.

(15 points) Using the above definition of Cauchy sequence, the lemma and the Triangle Inequality, prove the sequence $a_n = \dfrac{1}{n}$ is Cauchy.

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(15 points) Using the above definition of Cauchy sequence, the lemma and the Triangle Inequality, prove the sequence $a_n = \frac{1}{n}$ is Cauchy.

Solution: $\forall \epsilon \in \mathbb{Q}^+$ we need to show that $\exists M_\epsilon \in \mathbb{N}^+$ such that $m, n \geq M_\epsilon$ implies $\left|\frac{1}{m} - \frac{1}{n}\right| < \epsilon$. By the Triangle Inequality we know that

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \left|\frac{1}{m}\right| + \left|-\frac{1}{n}\right| = \frac{1}{m} + \frac{1}{n}.$$

If we choose M_ϵ so that $m, n \geq M_\epsilon$ implies $\frac{1}{m} < \frac{\epsilon}{2}$ and $\frac{1}{n} < \frac{\epsilon}{2}$, then we will get

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The condition $\frac{1}{m} < \frac{\epsilon}{2}$ is equivalent to $\frac{2}{\epsilon} < m$ and $\frac{1}{n} < \frac{\epsilon}{2}$ is equivalent to $\frac{2}{\epsilon} < n$. Using the lemma with $x = \frac{2}{\epsilon}$, there is some $N_x \in \mathbb{N}^+$ such that $\frac{2}{\epsilon} < N_x$, so if $m, n \geq M_\epsilon = N_x$ we have $\frac{2}{\epsilon} < M_x \leq m, n$ as required.
