NAME (Printed): ____

Math 330-3 Number Systems Fall 2022 Quiz 8 Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS.

$\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$ and $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r > 0\}$

Def: For any sequence of rational numbers a_n, $n \in \mathbb{N}^+$, we say $\lim_{n\to\infty} a_n = L$ when $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $n \geq M_\epsilon$ then $|a_n - L| < \epsilon$.

Def: We say a sequence of rational numbers a_n, $n \in \mathbb{N}^+$, is **Cauchy** when $\forall \epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $m, n \geq M_\epsilon$ then $|a_m - a_n| < \epsilon$.

Lemma: $\forall x \in \mathbb{R}$, $\exists N_x \in \mathbb{N}^+$ such that $x < N_x$.

Theorem (Triangle Inequality in $\mathbb{Q}$) For any $a, b \in \mathbb{Q}$ we have $|a + b| \leq |a| + |b|$.

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1. (10 points) Suppose a_n is a rational Cauchy sequence and $r \in \mathbb{Q}^+$. Prove that the rational sequence ra_n is Cauchy.

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2. (10 points) Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ for two rational sequences. Prove that $\lim_{n\to\infty} c_n = L + M$ for the sequence $c_n = a_n + b_n$.

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Def: For any sequence of rational numbers a_n, $n \in \mathbb{N}^+$, we say $\lim_{n\to\infty} a_n = L$ when $\forall\epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $n \geq M_\epsilon$ then $|a_n - L| < \epsilon$.

Def: We say a sequence of rational numbers a_n, $n \in \mathbb{N}^+$, is **Cauchy** when $\forall\epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $m, n \geq M_\epsilon$ then $|a_m - a_n| < \epsilon$.

Lemma: $\forall x \in \mathbb{R}$, $\exists N_x \in \mathbb{N}^+$ such that $x < N_x$.

Theorem (Triangle Inequality in $\mathbb{Q}$) For any $a, b \in \mathbb{Q}$ we have $|a + b| \leq |a| + |b|$.

(a) (10 points) Suppose a_n is a rational Cauchy sequence and $r \in \mathbb{Q}^+$. Prove that the rational sequence ra_n is Cauchy.

Solution: $\forall\epsilon \in \mathbb{Q}^+$ we need to show that $\exists M_\epsilon \in \mathbb{N}^+$ such that $m, n \geq M_\epsilon$ implies $|ra_m - ra_n| < \epsilon$. We know that $|ra_m - ra_n| = r|a_m - a_n|$ so we want $r|a_m - a_n| < \epsilon$, which is equivalent to $|a_m - a_n| < \frac{\epsilon}{r}$. Since a_n is Cauchy, using $\epsilon_1 = \frac{\epsilon}{r} \in \mathbb{Q}^+$ in the definition, $\exists M_{\epsilon_1} \in \mathbb{N}^+$ such that $m, n \geq M_{\epsilon_1}$ implies $|a_m - a_n| < \epsilon_1 = \frac{\epsilon}{r}$ giving the result.

(b) (10 points) Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ for two rational sequences. Prove that $\lim_{n\to\infty} c_n = L + M$ for the sequence $c_n = a_n + b_n$.

Solution: We are given that $\forall\epsilon_1 \in \mathbb{Q}^+$, $\exists M_{\epsilon_1} \in \mathbb{N}^+$ such that if $n \geq M_{\epsilon_1}$ then $|a_n - L| < \epsilon_1$ and $\forall\epsilon_2 \in \mathbb{Q}^+$, $\exists M_{\epsilon_2} \in \mathbb{N}^+$ such that if $n \geq M_{\epsilon_2}$ then $|b_n - M| < \epsilon_2$. We wish to show that $\forall\epsilon \in \mathbb{Q}^+$, $\exists M_\epsilon \in \mathbb{N}^+$ such that if $n \geq M_\epsilon$ then $|(a_n + b_n) - (L + M)| < \epsilon$. The last inequality can be written as $|(a_n - L) + (b_n - M)| < \epsilon$. From the Triangle Inequality we know that $|(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M|$ so if we use $\epsilon_1 = \epsilon/2 = \epsilon_2$ then there exist $M_{\epsilon_1} \in \mathbb{N}^+$ and $M_{\epsilon_2} \in \mathbb{N}^+$ such that if $n \geq M_{\epsilon_1}$ then $|a_n - L| < \epsilon_1$ and if $n \geq M_{\epsilon_2}$ then $|b_n - M| < \epsilon_2$. So if we let $M_\epsilon = Max(M_{\epsilon_1}, M_{\epsilon_2})$, then for $n \geq M_\epsilon$ we have

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \epsilon_1 + \epsilon_2 = \epsilon$$

does the job.
