NAME (Printed): ______________________________

Math 330-3 Number Systems Fall 2022 Quiz 9 Feingold

SHOW ALL WORK NECESSARY TO JUSTIFY YOUR ANSWERS.

You may use basic properties of bijections that we have already proven.

For S and T any two sets, define a relation $S \sim T$ when there exists **some bijection** $f : S \to T$.

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1. (9 points) Prove that this relation has the properties of an equivalence relation: reflexive, symmetric and transitive.

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2. (6 points) Prove that $\mathbb{Z} \sim 2\mathbb{Z}$.

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1. (9 points) Prove that this relation has the properties of an equivalence relation: reflexive, symmetric and transitive.

Solution:

Reflexive: $S \sim S$ because the identity map $I_S : S \to S$ defined by $I_S(x) = x$ for all $x \in S$, is bijective.

Symmetric: If $S \sim T$ then there exists a bijection $f : S \to T$, so f has an inverse $f^{-1} : T \to S$ which is also bijective, so $T \sim S$.

Transitive: If $S \sim T$, and $T \sim R$, then there are bijections $f : S \to T$ and $g : T \to R$. Their composition $g \circ f : S \to R$ is also bijective, so $S \sim R$.

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2. (6 points) Prove that $\mathbb{Z} \sim 2\mathbb{Z}$.

Solution: Let $f : \mathbb{Z} \to 2\mathbb{Z}$ be the map defined by $f(n) = 2n$. Then f is injective because $f(m) = f(n)$ means $2m = 2n$ so $2(m - n) = 0$ so $m - n = 0$ so $m = n$. Also, f is surjective because $2\mathbb{Z} = \{2n \in \mathbb{Z} \mid n \in \mathbb{Z}\} = Image(f)$. Since f is a bijection, we have shown that $\mathbb{Z} \sim 2\mathbb{Z}$.
