Here is a question about dual bases and how the transition matrix ${}_SP_T$ for a pair of bases for V is related to the transition matrix ${}_{S^*}Q_{T^*}$ for the corresponding dual bases. It is an example of a general theorem we did not have time to discuss in class, but uses only the basic definitions we have covered.

Let $V = \mathbf{R}^3$ with standard basis $S = \{e_1, e_2, e_3\}$ and another basis $T = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. Let V^* be the dual space of V with basis $S^* = \{f_1, f_2, f_3\}$ dual to S such that $f_j(e_i) = \delta_{ij}$. There is another basis of V^*, $T^* = \{g_1, g_2, g_3\}$ dual to T such that $g_j(v_i) = \delta_{ij}$. Find T^* explicitly by expressing $g_j = a_{1j}f_1 + a_{2j}f_2 + a_{3j}f_3$ for $j = 1, 2, 3$, that is, find the transition matrix ${}_{S^*}Q_{T^*} = [a_{ij}]$. How is it related to the transition matrix ${}_SP_T$ from T to S?

Solution: Use the relations $\delta_{ij} = g_j(v_i) = a_{1j}f_1(v_i) + a_{2j}f_2(v_i) + a_{3j}f_3(v_i)$ for $i, j = 1, 2, 3$. First write $v_1 = 1e_1 + 2e_2 + 1e_3$, $v_2 = 1e_1 + 1e_2$ and $v_3 = 1e_1$, so that

$$\begin{aligned} f_1(v_1) &= 1, & f_2(v_1) &= 2, & f_3(v_1) &= 1, \\ f_1(v_2) &= 1, & f_2(v_2) &= 1, & f_3(v_2) &= 0, \\ f_1(v_3) &= 1, & f_2(v_3) &= 0, & f_3(v_3) &= 0 \end{aligned}$$

and then we get the nine equations

$$\begin{aligned} 1a_{11} + 2a_{21} + 1a_{31} &= 1, & 1a_{11} + 1a_{21} &= 0, & 1a_{11} &= 0 \\ 1a_{12} + 2a_{22} + 1a_{32} &= 0, & 1a_{12} + 1a_{22} &= 1, & 1a_{12} &= 0 \\ 1a_{13} + 2a_{23} + 1a_{33} &= 0, & 1a_{13} + 1a_{23} &= 0, & 1a_{13} &= 1 \end{aligned}$$

which are equivalent to the single matrix equation

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which tells us that

$${}_{S^*}Q_{T^*} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

so we find that

$$g_1 = f_3, \qquad g_2 = f_2 - 2f_3, \qquad g_3 = f_1 - f_2 + f_3$$

This says that

$${}_{S^*}Q_{T^*} = \left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{Tr} \right)^{-1} = (({}_SP_T)^{Tr})^{-1} = (({}_SP_T)^{-1})^{Tr} = ({}_TP_S)^{Tr}.$$
