Math 507 Linear Algebra and Matrix Theory Fall 2010 Exam 1 Feingold

(1) (20 Points) Let $V = \{f : \mathbf{R} \to \mathbf{R} \mid f \text{ is differentiable}\}$, so V is a vector space over the field of reals, $\mathbf{R}$. Let

$$W = \{f \in V \mid f(0) = 0 \text{ and } f''(1) + f'(1) = 0\}.$$

Prove that W is a subspace of V.

(2) (30 Points) Answer (with brief justification) each question separately.

(a) If $S \subseteq T \subseteq V$ and S is dependent, what is the most you can say about T?
(b) If $S \subseteq T \subseteq V$ and T is independent, what is the most you can say about S?
(c) If $T = \{w_1, w_2, \ldots, w_m\} \subseteq W$ and $w \in \langle T \rangle$, what is the most you can say about $T \cup \{w\} = \{w_1, \ldots, w_m, w\}$?
(d) If $S = \{v_1, \ldots, v_n\} \subseteq V$ is independent and $v \in V$ but $v \notin \langle S \rangle$, what is the most you can you say about $S \cup \{v\} = \{v_1, \ldots, v_n, v\}$?
(e) If $A \in F_n^n$ is row equivalent to the identity matrix, what is the most you can say about the associated linear map L_A?
(f) Let $A \in \mathbf{F}_n^m$ with $m < n$. What is the most you can say about the number of free variables in the solutions to the linear system $AX = 0$?

(3) (20 Points) Let S be the following subset of $\mathbf{R}_2^2$,

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 1 & -4 \end{bmatrix} \right\}$$

and let $W = \langle S \rangle$ be the span of S.

(a) Show that S is dependent and find the nontrivial dependence relations which allow redundant vectors to be removed.
(b) Find conditions on a, b, c, d such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$. Is $W = \mathbf{R}_2^2$?
(c) Find a basis for W.

(4) (10 points) Answer each question separately.

(a) Suppose $0_9^4 \neq A \in F_9^4$ and $L_A : F^9 \to F^4$ is the linear map $L_A(X) = AX$. What are the possibilities for the number of vectors in a basis for $\text{Ker}(L_A)$?
(b) Suppose $A, B, C \in F_n^n$ are invertible matrices. Write the inverse of their product, $(ABC)^{-1}$ in terms of the inverses of the three matrices.

(5) (20 Points) Let $L_A : \mathbf{C}^4 \to \mathbf{C}^3$ be the linear map given by $L_A(X) = AX$ where

$$A = \begin{bmatrix} 1 & -1 & \mathbf{i} & 1+\mathbf{i} \\ \mathbf{i} & -\mathbf{i} & -1 & -1+\mathbf{i} \\ 1+\mathbf{i} & -1-\mathbf{i} & -1+\mathbf{i} & 2\mathbf{i} \end{bmatrix} \in \mathbf{C}^3_4$$

(a) (5 points) Find the set of all vectors in $\mathrm{Ker}(L_A)$.
(b) (5 points) Find a basis for $\mathrm{Ker}(L_A)$.
(c) (2 points) Is L_A one-to one? **Explain why!**
(d) (6 points) Find a basis for $\mathrm{Range}(L_A)$.
(e) (2 point) Is L_A onto? **Explain why!**

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1. (20 Points)Let $V = \{f : \mathbf{R} \to \mathbf{R} \mid f \text{ is differentiable}\}$, so V is a vector space over the field of reals, $\mathbf{R}$. Let

$$W = \{f \in V \mid f(0) = 0 \text{ and } f''(1) + f'(1) = 0\}.$$

   (a) Prove that W is a subspace of V.

SOLUTION: To show W is a subspace, show three facts: the zero function is in W and that W is closed under addition and scalar multiplication. The zero function defined by $\theta(t) = 0$ for all $t \in \mathbf{R}$, is in W because $\theta(0) = 0$, and $\theta'(t) = \theta''(t) = \theta(t)$, so $\theta''(1) + \theta'(1) = 0$. If $f, g \in W$, then $f + g \in W$ because $(f + g)(t) = f(t) + g(t)$ so $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$ and $(f + g)''(1) + (f + g)'(1) = f''(1) + g''(1) + f'(1) + g'(1) = f''(1) + f'(1) + g''(1) + g'(1) = 0 + 0 = 0$. If $f \in W$ and $b \in \mathbf{F}$ then $bf \in W$ since $(bf)(0) = b(f(0)) = b(0) = 0$ and $(bf)''(1) + (bf)'(1) = b(f''(1) + f'(1)) = b(0) = 0$.

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2. (30 points)
   (a) T is also a dependent set. (A set containing a dependent set is dependent.)
   (b) S is independent. (A subset of an independent set is independent.)
   (c) $T \cup \{w\}$ is dependent. (Last vector is a linear combo of previous vectors.)
   (d) $S \cup \{v\}$ is independent. (No vector is a linear combo of previous vectors.)
   (e) A is invertible, so L_A is invertible, bijective, and has inverse $L_{A^{-1}}$.
   (f) Since $m < n$ the linear system has more variables than equations, so it must have nontrivial solutions. When the augmented matrix is row reduced, it can have at most m leading ones, so there are at least $n-m$ columns without leading ones, giving at least $n - m$ free variables in the solution space.

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3. (20 points) (a) Let the 5 vectors of S be denoted $v_1, \cdots, v_5$. Determine if $\sum_{i=1}^{5} a_i v_i = \theta$ has nontrivial solutions. Reduce

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 3 & -2 & 5 & 0 \\ -1 & -2 & 2 & -2 & 1 & 0 \\ 1 & 2 & -4 & 3 & -4 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -s \\ x_2 = -\frac{1}{2}r - \frac{1}{2}s \\ x_3 = \frac{1}{2}r - \frac{3}{2}s \\ x_4 = r \in \mathbf{R} \\ x_5 = s \in \mathbf{R} \end{array}$$

which has nontrivial solutions so S is dependent. Each free variable gives a dependence relation. When $r = 1$ and $s = 0$, we get $-\frac{1}{2}v_2 + \frac{1}{2}v_3 + v_4 = \theta$ and when $r = 0$ and $s = 1$ we get $-v_1 - \frac{1}{2}v_2 - \frac{3}{2}v_3 + v_5 = \theta$. These allow the redundant vectors v_4 and v_5 to be expressed as linear combinations of v_1, v_2 and v_3, so S is spanned by just those three vectors.

(b) Since $W = \langle\{v_1, v_2, v_3\}\rangle$, a matrix is in this span when the following system is consistent:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 1 & -1 & 3 & b \\ -1 & -2 & 2 & c \\ 1 & 2 & -4 & d \end{array}\right] \text{ reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2a+c \\ 0 & 1 & 0 & -a-\frac{3}{2}c-\frac{1}{2}d \\ 0 & 0 & 1 & \frac{1}{2}c-\frac{1}{2}d \\ 0 & 0 & 0 & -3a+b-c+d \end{array}\right]$$

which is consistent iff $0 = -3a + b - c + d$. This is the condition required, and it is not true for all matrices so $W \neq \mathbf{R}_2^2$.

(c) From part (a), after removing the redundant vectors from S, we have $\{v_1, v_2, v_3\}$ is a basis for W. Another possible answer is obtained by using the condition from part (b), which expresses any element of W as

$$\begin{bmatrix} a & b \\ c & 3a-b+c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

so a basis of W is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$.

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4. (10 points) (a) Since $0_9^4 \neq A \in F_9^4$, A row reduces to a RREF with r leading 1's, and $1 \leq r \leq 4$ so $9 - r \in \{5, 6, 7, 8\}$ is number of possible free variables.
(b) By a theorem proved in class, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

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5. (20 points) To find Ker(L), row reduce

$$\left[\begin{array}{cccc|c} 1 & -1 & \mathbf{i} & 1+\mathbf{i} & 0 \\ \mathbf{i} & -\mathbf{i} & -1 & -1+\mathbf{i} & 0 \\ 1+\mathbf{i} & -1-\mathbf{i} & -1+\mathbf{i} & 2\mathbf{i} & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & -1 & \mathbf{i} & 1+\mathbf{i} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = r - \mathbf{i}s - (1+\mathbf{i})t \\ x_2 = r \in \mathbf{C} \\ x_3 = s \in \mathbf{C} \\ x_4 = t \in \mathbf{C} \end{array}.$$

(a) (5 points) $\operatorname{Ker}(L_A) = \left\{ \begin{bmatrix} r - \mathbf{i}s - (1+\mathbf{i})t \\ r \\ s \\ t \end{bmatrix} \in \mathbf{C}^4 \;\middle|\; r, s, t \in \mathbf{C} \right\}$.

(b) (5 points) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\mathbf{i} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1-\mathbf{i} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of Ker(L_A).

(c) (2 points) L_A is not one-to-one since Ker(L_A) is nontrivial.

(d) (6 points) Range(L_A) is the span of the columns of A, but the three free variables in Ker(L_A) mean the last three columns of A are redundant vectors, so a basis for Range(L) is column one of A.

(e) (2 points) L_A is not onto, Range(L_A) $\neq \mathbf{C}^3$.