Math 404 Advanced Linear Algebra Spring 2019 Practice Exam 2 Feingold

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1. (25 points) Let $L : \mathbf{R}^2_2 \to \mathbf{R}^2$ be given by $L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - b + 2c - 3d \\ -a + 3b + 5c + d \end{bmatrix}$.
Let S be the standard basis of $\mathbf{R}^2_2$ and let T be the standard basis of $\mathbf{R}^2$. Let other ordered bases be

$$S' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\} \text{ and } T' = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

   (a) (5 pts) Find the matrix ${}_T[L]_S$ representing L from S to T.

   (b) (8 pts) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)

   (c) (12 pts) Find the transition matrices ${}_SP_{S'}$ and ${}_TQ_{T'}$ and show that ${}_{T'}[L]_{S'} = ({}_TQ_{T'})^{-1}\ {}_T[L]_S\ ({}_SP_{S'})$.

2. (20 points, 5 points each) Answer each of the following questions separately.

   (a) Suppose U_1, U_2 and U_3 are subspaces of V. Under what conditions will the sum $U_1 + U_2 + U_3$ be a **direct sum**? (Do not just give the definition.)

   (b) Let $L : V \to V$, let $\theta \neq v \in V$ be an eigenvector for L with eigenvalue $\lambda \in \mathbf{F}$. Show that v is an eigenvector for $L^3 = L \circ L \circ L$ with eigenvalue λ^3.

   (c) Let $A, B \in \mathbf{F}^n_n$ be invertible matrices satisfying the relation $ABA^{-1} = B^{-1}$. What is the **most** you can say about $\det(A)$? What is the **most** you can say about $\det(B)$?

   (d) Let $A \in \mathbf{F}^n_n$ have n **distinct** eigenvalues $\lambda_1, \cdots, \lambda_n$. What is the **most** you can say about the characteristic polynomial $p_A(t)$, the minimal polynomial $m_A(t)$, and whether or not A is diagonalizable?

3. (35 Points) Let $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ represent a transformation $L : \mathbf{R}^4 \to \mathbf{R}^4$ with respect to the standard basis S of $\mathbf{R}^4$.

   (a) (10 points) Find the **characteristic polynomial**, $p_A(t) = \det(A - tI_4)$, and find the **eigenvalues** of A, and their **algebraic multiplicities**.

   (b) (20 points) Can A can be diagonalized? **If not, give reasons why**. If it can, **find a basis** T of $\mathbf{R}^4$ and a **diagonal matrix** $D = {}_TD_T$ representing L from T to T. Also **find the transition matrix** $P = {}_SP_T$, **find the inverse transition matrix** $P^{-1} = {}_TP_S$, and **check that** $D = P^{-1}AP$.

   (c) (5 points) Find the **minimum polynomial**, $m_A(t)$, and **check that** $m_A(A)$ is the **zero matrix**.

4. (20 Points, 5 points each) Answer each of the following questions separately.

(a) Find $\det \begin{bmatrix} 4 & 6 & 6 & 8 \\ 3 & -9 & 6 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(b) If $A^t = -A$ for a matrix $A \in \mathbf{R}^n_n$ where n is odd, what is the **most** you can say about $det(A)$?

(c) Let $L : V \to V$ have **distinct** eigenvalues $\lambda_1, \cdots, \lambda_r$, and for $1 \leq i \leq r$ let $T_i = \{v_{ij} \mid 1 \leq j \leq g_i\}$ be a **basis** of the λ_i eigenspace, L_{λ_i}. What is the **most** you can say about the union of all these sets $T = T_1 \cup \cdots \cup T_r$?

(d) Let $T = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a basis of $\mathbf{R}^3$ and let $v = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix} \in \mathbf{R}^3$.
Find the coordinate vector $[v]_T$.

1. (25 Pts)

(a) (5 Pts) ${}_T[L]_S = \begin{bmatrix} 1 & -1 & 2 & -3 \\ -1 & 3 & 5 & 1 \end{bmatrix}$ is easy to get since S and T are standard.

$$L\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ L\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},\ L\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix},\ L\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

$[T \mid L(S)] = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 & 5 & 1 \\ \multicolumn{2}{c|}{T} & \multicolumn{4}{c}{L(S)} \end{array}\right]$ is in RREF, so the right side is ${}_T[L]_S$.

(b) (8 Pts)

$$L\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix},\ L\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix},\ L\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix},\ L\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

Row reduce $\left[\begin{array}{cc|cccc} 3 & 5 & -2 & 1 & -1 & -3 \\ 1 & 2 & 0 & 8 & 6 & 3 \\ \multicolumn{2}{c|}{T'} & \multicolumn{4}{c}{L(S')} \end{array}\right]$ to $\left[\begin{array}{cc|cccc} 1 & 0 & -4 & -38 & -32 & -21 \\ 0 & 1 & 2 & 23 & 19 & 12 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{4}{c}{{}_{T'}[L]_{S'}} \end{array}\right]$

(c) (12 Pts) ${}_SP_{S'} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ since S and T are the standard bases.

To get ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$, reduce

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \multicolumn{2}{c|}{T'} & \multicolumn{2}{c}{T} \end{array}\right] \text{ to } \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{2}{c}{{}_{T'}Q_T} \end{array}\right]$$

$$({}_TQ_{T'})^{-1}{}_T[L]_S({}_SP_{S'}) = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & -3 \\ -1 & 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 7 & -17 & -21 & -11 \\ -4 & 10 & 13 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -38 & -32 & -21 \\ 2 & 23 & 19 & 12 \end{bmatrix} = {}_{T'}[L]_{S'} \text{ checks.}$$

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2. (20 points, 5 points each)
   (a) The sum $U_1 + U_2 + U_3$ is a **direct sum** when $U_1 \cap (U_2 + U_3) = \{\theta\}$, $U_2 \cap (U_1 + U_3) = \{\theta\}$ and $U_3 \cap (U_1 + U_2) = \{\theta\}$.
   (b) $L^3(v) = (L \circ L \circ L)(v) = L(L(L(v))) = L(L(\lambda v) = \lambda L(L(v)) = \lambda L(\lambda v) = \lambda^2 L(v) = \lambda^2 \lambda v = \lambda^3 v$.
   (c) Since $\det(ABA^{-1}) = \det(B^{-1})$ we get from the multiplicative property of det that $\det(A)\det(B)\det(A)^{-1} = \det(B)^{-1}$ so $\det(B)^2 = 1$ which gives $\det(B) = \pm 1$. Since A is invertible, all we can say is that $\det(A) \neq 0$.
   (d) We can say that $p_A(t) = (-1)^n \prod_{i=1}^n (t - \lambda_i) = (-1)^n m_A(t)$ and A is diagonalizable, is similar to the diagonal matrix with $\lambda_1, \cdots, \lambda_n$ on the diagonal.

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3. (35 points)
   (a) (10 points) The characteristic polynomial is $p_A(t) = \det(A - tI_4) =$

$$\det \begin{bmatrix} -t & 1 & 1 & 1 \\ 1 & -t & 1 & 1 \\ 1 & 1 & -t & 1 \\ 1 & 1 & 1 & -t \end{bmatrix} = \det \begin{bmatrix} -t-1 & 0 & 0 & t+1 \\ 0 & -t-1 & 0 & t+1 \\ 0 & 0 & -t-1 & t+1 \\ 1 & 1 & 1 & -t \end{bmatrix}$$

$$= (t+1)^3 \det \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -t \end{bmatrix} = (t+1)^3 \det \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -t+3 \end{bmatrix}$$

$$= (t+1)^3 \, (t-3).$$

So the eigenvalues are $\lambda_1 = -1$ with algebraic multiplicity $k_1 = 3$ and $\lambda_2 = 3$ with algebraic multiplicity $k_2 = 1$.

   (b) (20 points) Check the $\lambda_1 = -1$ eigenspace first since the algebraic multiplicity $k_1 = 3$. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = -1$ to $A - tI_4$. Row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = -r - s - t \\ x_2 = r \in \mathbf{R} \\ x_3 = s \in \mathbf{R} \\ x_4 = t \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} -r-s-t \\ r \\ s \\ t \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r, s, t \in \mathbf{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

with three vectors. Since there will be one more independent eigenvector from the other eigenvalue, we will have the necessary four eigenvectors to form a basis for $\mathbf{R}^4$, so this A is diagonalizable.

Now find the $\lambda_2 = 3$ eigenspace. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = 3$ to $A - tI_4$. Row reduce

$$\left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 0 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -3 & 1 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r \\ x_2 = r \\ x_3 = r \\ x_4 = r \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r \in \mathbf{R} \right\} \qquad \text{has basis} \qquad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Therefore, $T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, ${}_TD_T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and

$P = {}_SP_T = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = {}_TP_S = \frac{1}{4} \begin{bmatrix} -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. We check:

$$P^{-1}AP = \frac{1}{4} \begin{bmatrix} -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 3 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = {}_TD_T.$$

(c) (5 points) The minimum polynomial $m_A(t)$ divides the characteristic polynomial, $p_A(t) = (t+1)^3(t-3)$ and has the same roots, so the only options are $(t+1)^i(t-3)$ for $i = 1, 2, 3$. Checking $i = 1$ first we find $(A + 1I_4)(A - 3I_4) =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is the zero matrix, so $m_A(t) = (t+1)(t-3)$.

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4. (20 Points, 5 points each) Answer each of the following questions separately.

(a)

$$\det\begin{bmatrix}4&6&6&8\\3&-9&6&3\\2&1&1&0\\1&2&3&4\end{bmatrix} = (2)(3)\det\begin{bmatrix}2&3&3&4\\1&-3&2&1\\2&1&1&0\\1&2&3&4\end{bmatrix}$$

$$= (6)\det\begin{bmatrix}0&-1&-3&-4\\0&-5&-1&-3\\0&-3&-5&-8\\1&2&3&4\end{bmatrix} = -(6)\det\begin{bmatrix}1&2&3&4\\0&-5&-1&-3\\0&-3&-5&-8\\0&-1&-3&-4\end{bmatrix}$$

$$= (6)\det\begin{bmatrix}1&2&3&4\\0&-1&-3&-4\\0&-3&-5&-8\\0&-5&-1&-3\end{bmatrix} = (6)(-1)(-1)\det\begin{bmatrix}1&2&3&4\\0&-1&-3&-4\\0&3&5&8\\0&5&1&3\end{bmatrix}$$

$$= (6)\det\begin{bmatrix}1&2&3&4\\0&-1&-3&-4\\0&0&-4&-4\\0&0&-14&-17\end{bmatrix} = (6)(-4)\det\begin{bmatrix}1&2&3&4\\0&-1&-3&-4\\0&0&1&1\\0&0&0&-3\end{bmatrix}$$

$$=(6)(-4)(1)(-1)(1)(-3) = -72.$$

(b) If $A^t = -A$ for $A \in \mathbf{R}^n_n$ where n is odd, then $det(A) = det(A^t) = det(-A) = (-1)^n det(A) = -det(A)$, so we can say that $det(A) = 0$.

(c) We can say that $T = T_1 \cup \cdots \cup T_r$ is **linearly independent**.

(d) $[v]_T = \begin{bmatrix}-1\\10\\0\end{bmatrix}$ since $\left[\begin{array}{ccc|c}1&1&0&9\\2&1&1&8\\3&1&1&7\end{array}\right]$ reduces to $\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&10\\0&0&1&0\end{array}\right]$.

As a check:

$$-1\begin{bmatrix}1\\2\\3\end{bmatrix} + 10\begin{bmatrix}1\\1\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}9\\8\\7\end{bmatrix}.$$