Math 404 Advanced Linear Algebra Spring 2019 Practice Exam 3 Feingold

1. (20 Points) Suppose V is a real inner product space with inner product $\langle \cdot, \cdot \rangle$.
   (a) (5 Pts) State the Cauchy-Schwarz inequality for V.
   (b) (5 Pts) If $T = \{v_1, \cdots, v_m\}$ is an **orthogonal** set of nonzero vectors in V, prove that T is **independent**.
   (c) (5 Pts) Let $S = \{v_1, \cdots, v_n\}$ be an **orthonormal basis** of V, so for any $v \in V$ we can write $v = \sum_{i=1}^{n} c_i v_i$. Use the inner product to give a formula for the Fourier coefficients c_i.
   (d) (5 Pts) Let $T = \{v_1, \cdots, v_m\}$ be an **orthogonal basis** for a subspace $W \leq V$. Write the formula for the **projection** $Proj_W(v)$ of any vector $v \in V$ onto W.

2. (15 Points) Let $L = L_C : \mathbf{R}^2 \to \mathbf{R}^2$ for companion matrix $C = C((t-4)^2) = \begin{bmatrix} 0 & -16 \\ 1 & 8 \end{bmatrix}$. The characteristic and minimal polynomials for C are both equal to $(t-4)^2$, so the Jordan form matrix similar to C must be $J = J(4,2) = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$. Find a **Jordan basis** $T = \{v_1, v_2\}$ for $\mathbf{R}^2$ such that ${}_T[L]_T = J$.

3. (20 Points) Let $M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
   (a) (10 points) Show that M is **positive definite**.
   (b) (10 points) Let $M = [\langle e_i, e_j \rangle]$ represent an inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^2$ with respect to the standard basis $S = \{e_1, e_2\}$. Find the cosine of the angle $\theta_{v,w}$ between v and w.

4. (25 points) Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be $L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y - z \\ y + z \\ x - y \end{bmatrix}$, let $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and let $T = \{v, L(v), L^2(v)\}$.
   (a) Find T and show it is independent, so it is a basis of $\mathbf{R}^3$.
   (b) Find $L^3(v)$ and express it as a linear combination of the vectors in T.
   (c) Using the answers to parts (a) and (b) find the companion matrix $C = {}_T[L]_T$ that represents L from T to T and find its **characteristic** polynomial, $\Delta_C(t)$.

5. (20 Points) Answer each of the following questions separately.
   (a) Suppose A has **characteristic** and **minimal** polynomials $\Delta_A(t) = \det(tI_{11} - A) = (t-2)^6(t-7)^5$ and $m_A(t) = (t-2)^3(t-7)^4$. Find all the possible **Jordan Canonical Form** matrices J that could be similar to A, but not similar to each other. In each case, give g_2 and g_7, the **geometric multiplicities** for each eigenvalue.
   (b) Suppose $A \in \mathbf{R}^{14}_{14}$ has characteristic and minimal polynomials $\Delta_A(t) = (t^2+2)^4(t^2+t+2)^3$ and $m_A(t) = (t^2+2)^2(t^2+t+2)^2$. Find all the possible **Rational Canonical Form** matrices that could be similar to A, but not similar to each other.

1. (20 Points) Suppose V is a real inner product space with inner product $\langle \cdot, \cdot \rangle$.

(a) (5 Pts) State the Cauchy-Schwarz inequality for V.

Solution: For any $u, v \in V$ we have $\langle u, v\rangle^2 \leq \langle u, u\rangle \ \langle v, v\rangle$.

(b) (5 Pts) If $T = \{v_1, \cdots, v_m\}$ is an **orthogonal** set of nonzero vectors in V, prove that T is independent.

Solution: If $\sum_{i=1}^m a_i v_i = \theta$ then taking the inner product of both sides with v_j gives $\sum_{i=1}^m a_i \langle v_i, v_j\rangle = \langle \theta, v_j\rangle = 0$, but since $\langle v_i, v_j\rangle = 0$ for $i \neq j$, this says $a_j \langle v_j, v_j\rangle = 0$. Since $v_j \neq \theta$, we know $\langle v_j, v_j\rangle \neq 0$, so $a_j = 0$ is true for each $1 \leq j \leq n$. This shows that T is independent.

(c) (5 Pts) Let $S = \{v_1, \cdots, v_n\}$ be an **orthonormal basis** of V, so for any $v \in V$ we can write $v = \sum_{i=1}^n c_i v_i$. Use the inner product to give a formula for the Fourier coefficients c_i.

Solution: For each $1 \leq j \leq n$ we have $\langle v, v_j\rangle = \sum_{i=1}^n c_i \langle v_i, v_j\rangle = \sum_{i=1}^n c_i \delta_{ij} = c_j$, so the formula is $c_j = \langle v, v_j\rangle$.

(d) (5 Pts) Let $T = \{v_1, \cdots, v_m\}$ be an **orthogonal basis** for a subspace $W \leq V$. Write the formula for the **projection** $Proj_W(v)$ of any vector $v \in V$ onto W.

Solution: The formula for the projection is $Proj_W(v) = \displaystyle\sum_{i=1}^m \frac{\langle v, v_i\rangle}{\langle v_i, v_i\rangle} v_i$.

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2. (15 Points) Let $L = L_C : \mathbf{R}^2 \to \mathbf{R}^2$ for companion matrix $C = C((t-4)^2) = \begin{bmatrix} 0 & -16 \\ 1 & 8 \end{bmatrix}$. The characteristic and minimal polynomials for C are both equal to $(t-4)^2$, so the Jordan form matrix similar to C must be $J = J(4,2) = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$. Find a **Jordan basis** $T = \{v_1, v_2\}$ for $\mathbf{R}^2$ such that ${}_T[L]_T = J$.

Solution: The only eigenvalue for C is $\lambda_1 = 4$ and the corresponding eigenspace C_4 is the null-space of $C - 4I_2 = \begin{bmatrix} -4 & -16 \\ 1 & 4 \end{bmatrix}$ which row reduces to $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$ so $C_4 = \left\{ \begin{bmatrix} -4r \\ r \end{bmatrix} \in \mathbf{R}^2 \mid r \in \mathbf{R} \right\}$ has basis $\left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$. Since C satisfies its minimal polynomial, we know that $(C - 4I_2)^2 = 0_2^2$ so the generalized eigenspace $C_4^{(2)} = \mathbf{R}^2$. We can choose for v_2 any vector in $\mathbf{R}^2$ which is not in C_4. The simplest choice is $v_2 = e_1$ the first standard basis vector, and then we want $v_1 = (C - 4I_2)v_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$. With $T = \left\{ v_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ we have $L(v_1) = Cv_1 = 4v_1$ and $L(v_2) = Cv_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_1 + 4v_2$ so that ${}_T[L]_T = J$ as required.

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3. (20 Points) Let $M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   (a) (10 points) Show that M is **positive definite**.

Solution: For any $u = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2$ we have

$$u^{Tr} M u = [\, x \quad y \,] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$2x^2 - 2xy + 2y^2 = x^2 + (x^2 - 2xy + y^2) + y^2 = x^2 + (x-y)^2 + y^2 \geq 0$$

since it is a sum of squares of real numbers. If $x^2 + (x-y)^2 + y^2 = 0$ then all terms are zero, so $x = x - y = y = 0$ giving u is the zero vector.

   (b) (10 points) Let $M = [\langle e_i, e_j \rangle]$ represent an inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^2$ with respect to the standard basis $S = \{e_1, e_2\}$. Find the cosine of the angle $\theta_{v,w}$ between v and w.

Solution: We have

$$\langle v, w \rangle = [v]_S^{Tr} M [w]_S = [\, 2 \quad -1 \,] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = [\, 5 \quad -4 \,] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -7,$$

$$\langle v, v \rangle = [\, 2 \quad -1 \,] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = [\, 5 \quad -4 \,] \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 14,$$

$$\langle w, w \rangle = [\, 1 \quad 3 \,] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = [\, -1 \quad 5 \,] \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 14 \text{ so}$$

$$\cos(\theta_{v,w}) = \frac{\langle v, w \rangle}{\sqrt{\langle v, v \rangle}\sqrt{\langle w, w \rangle}} = \frac{-7}{\sqrt{14}\sqrt{14}} = \frac{-7}{14} = \frac{-1}{2}.$$

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4. (25 points) We have $L : \mathbf{R}^3 \to \mathbf{R}^3$ by $L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y - z \\ y + z \\ x - y \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

and $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $T = \{v, L(v), L^2(v)\}$.

(a) $T = \left\{ v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, L(v) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, L^2(v) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. To show it is independent, show $x_1 v + x_2 L(v) + x_3 L^2(v) = \theta$ has only the trivial solution. We row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}$$

(b) $L^3(v) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. To express it as a linear combination of the vectors in T, solve $x_1 v + x_2 L(v) + x_3 L^2(v) = L^3(v)$. We row reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{ so } \begin{array}{r} x_1 = 4 \\ x_2 = -3 \\ x_3 = 2 \end{array}$$

This means $L^3(v) = 4v + -3L(v) + 2L^2(v)$.

(c) From the answers to parts (a) and (b), the companion matrix that represents L from T to T is

$$C = {}_T[L]_T = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$

and the characteristic polynomial is read off from the right column, or from the dependence relation found in part (b). It is

$$\Delta_L(t) = t^3 - 2t^2 + 3t - 4.$$

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5. (20 Points) Answer each of the following questions separately.

(a) (10 points) The **characteristic polynomial** is $\Delta_A(t) = (t-2)^6(t-7)^5$ and the **minimal polynomial** is $m_A(t) = (t-2)^3(t-7)^4$ so there are two eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 7$ with algebraic multiplicities $k_1 = 6$ and $k_2 = 5$. The powers in the minimal polynomial $m_1 = 3$ and $m_2 = 4$ tell the sizes of the largest Jordan blocks of each type. Let

$$B = J(2,3) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = J(2,2) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad D = J(7,4) = \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

Then there are only three possible Jordan canonical form matrices similar to A,

$$\begin{bmatrix} B & & & \\ & B & & \\ & & D & \\ & & & 7 \end{bmatrix}, \text{ or } \begin{bmatrix} B & & & & \\ & C & & & \\ & & 2 & & \\ & & & D & \\ & & & & 7 \end{bmatrix}, \text{ or } \begin{bmatrix} B & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & D & \\ & & & & & 7 \end{bmatrix}.$$

In the first case, there are two Jordan blocks for eigenvalue 2 so $g_2 = 2$. In the second case, there are three Jordan blocks for eigenvalue 2 so $g_2 = 3$. In the third case, there are four Jordan blocks for eigenvalue 2 so $g_2 = 4$. In all cases, there are two Jordan blocks for eigenvalue 7, so $g_7 = 2$.

(b) (10 points) Suppose $A \in \mathbf{R}^{14}_{14}$ has characteristic and minimal polynomials

$$\Delta_A(t) = (t^2+2)^4(t^2+t+2)^3 \qquad \text{and} \qquad m_A(t) = (t^2+2)^2(t^2+t+2)^2.$$

Find all the possible **Rational Canonical Form** matrices that could be similar to A, but not similar to each other.

Solution: Since there are two irreducible factors in $\Delta_A(t)$, there are two kinds of blocks in the RCF similar to A. The blocks coming from $(t^2+2)^4$ have a total size of 8, the degree of this factor, and the blocks coming from $(t^2+t+2)^3$ have a total size of 6 for the same reason. But because in $m_A(t)$ we have $(t^2+2)^2 = t^4+4t^2+4$, there must be at least one 4×4 companion matrix

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} = C((t^2+2)^2) = C(t^4+4t^2+4).$$

There could either be another block, C_1, or there could be two 2×2 companion matrices $C_2 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = C(t^2+2)$.

Since in $m_A(t)$ we have $(t^2+t+2)^2 = t^4+2t^3+5t^2+4t+4$, there must be one 4×4 companion matrix

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{bmatrix} = C((t^2+t+2)^2)$$

and there must also be one 2×2 companion matrix $C_4 = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} = C(t^2+t+2)$.

So there are two possible RCF matrices similar to A, either

$$diag[C_1, C_1, C_3, C_4] \qquad \text{or} \qquad diag[C_1, C_2, C_2, C_3, C_4].$$
