Math 404 Advanced Linear Algebra Spring 2023 Exam 1 Feingold

(1) (13 Points) Let $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$, so V is a vector space over the field of reals, $\mathbb{R}$, whose zero vector is the function $\theta(t) = 0$ for all $t \in \mathbb{R}$. Prove that $W = \{f \in V \mid f(1) = 0\}$ is a subspace of V.

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(2) (12 Points) Answer each question separately. **No justifications needed**.

(a) If $S \subseteq T \subseteq V$ and S is **dependent**, what is the **most** you can say about T?

(b) If $S \subseteq T \subseteq V$ and S is **independent**, what is the **most** you can say about T?

(c) Let $T = \{w_1, \ldots, w_m\} \subseteq W$, $w_{m+1} \in \langle T \rangle$ and $T_1 = T \cup \{w_{m+1}\} = \{w_1, \ldots, w_m, w_{m+1}\}$. What is the relationship between the **spans** $\langle T_1 \rangle$ and $\langle T \rangle$?

(d) Let $S = \{v_1, \ldots, v_k\} \subseteq V$ be **independent** and let $S_1 = S \cup \{v_{k+1}\} = \{v_1, \ldots, v_k, v_{k+1}\}$ for $v_{k+1} \in V$ with $v_{k+1} \notin \langle S \rangle$. What is $\dim(\langle S_1 \rangle)$?

(e) If $A \in \mathbb{F}^n_n$ is row equivalent to the identity matrix, write **at least 3 properties** of the associated linear map $L_A : \mathbb{F}^n \to \mathbb{F}^n$ defined by $L_A(X) = AX$.

(f) Let $A \in \mathbb{F}^m_n$ with $m < n$. Without knowing $Rank(A)$, what is the **most** you can say about the number of free variables in the solutions to the linear system $AX = 0$?

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(3) (15 Points) $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 4 \\ 1 & 2 & 0 & 2 & 5 \\ 1 & 3 & 1 & 0 & 6 \\ 1 & 4 & 0 & 2 & 7 \end{bmatrix}$ row reduces to $C = \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $L_A : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear map $L_A(X) = AX$.

(a) (3 Points) Find a **basis** for $Row(A)$, the span of the rows of A.

(b) (3 Points) Find a **basis** for $Ker(L_A) = Nul(A)$.

(c) (3 Points) Find a **basis** for $Range(L_A) = Col(A)$.

(d) (6 Points) Use answers to find the **dependence relations** among the columns of A.

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(4) (15 points) Answer each question separately. **No justifications needed**.

(a) For linear map $L : \mathbb{F}^9 \to \mathbb{F}^4$ what are all the possibilities for $\dim(Ker(L))$?

(b) For linear map $L : \mathbb{F}^2_3 \to \mathbb{F}^9$ what are all the possibilities for $\dim(Range(L))$?

(c) If $S = \{v_1, \cdots, v_n\}$ is a **basis** of V and $L : V \to W$, what is the **most** you can say about the set $L(S) = \{L(v_1), \cdots, L(v_n)\}$?

(d) If $T = \{w_1, \cdots, w_m\}$ is a **basis** of W what is the **most** you can say about the set of coordinate vectors $\{[w_1]_T, \cdots, [w_m]_T\}$?

(e) If $A \in \mathbb{F}^n_n$ and $AX = 0$ has only the trivial solution, then what is $Rank(A)$?

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(5) (15 Points) Let $L : \mathbb{R}_2^2 \to \mathbb{R}^2$ be the linear map given by $L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2b-c-2d \\ -a+b+c-d \end{bmatrix}$ and let S and T be the standard bases of $\mathbb{R}_2^2$ and $\mathbb{R}^2$, respectively. Let other ordered bases be $S' = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $T' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

(a) (2 points) Find the matrix ${}_T[L]_S$ representing L from S to T.

(b) (3 points) Find a basis for $\text{Ker}(L)$.

(c) (3 points) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' directly (without using transition matrices).

(d) (4 points) Find the transition matrices ${}_SP_{S'}$ and ${}_{T'}Q_T$.

(e) (3 point) Compute the product ${}_{T'}Q_T \; {}_T[L]_S \; {}_SP_{S'}$. Compare it to part (c).

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(6) (15 Points) Let $L : \mathbb{F}_n^n \to \mathbb{F}_n^n$ be defined by $L(A) = A + A^T$, where A^T means the transpose of A.

(a) (6 pts) Prove that L is a linear map.

(b) (4 pts) Find $Ker(L)$.

(c) (5 pts) Use the results in (a) and (b) prove that the set of all anti-symmetric matrices in $\mathbb{F}_n^n$ is a subspace.

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(7) (15 Points) Let $A \in \mathbb{F}_n^m$ be an $m \times n$ matrix of numbers from field $\mathbb{F}$ with $Rank(A) = r$. Let $\mathbf{0} \in \mathbb{F}^m$ be the $m \times 1$ zero matrix, and let $L_A : \mathbb{F}^n \to \mathbb{F}^m$ be the function associated with A defined as $L_A(X) = AX$. For each part below give the most accurate answer. **No justifications needed**.

(a) If $m > n$ then what property can you be sure the function L_A does **not have**?

(b) What is $\dim(Ker(L_A))$?

(c) What is $\dim(Range(L_A))$?

(d) What condition on r is equivalent to L_A being **injective**?

(e) What condition on r is equivalent to L_A being **surjective**?

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(1) (13 Points) Let $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$, so V is a vector space over the field of reals, $\mathbb{R}$, whose zero vector is the function $\theta(t) = 0$ for all $t \in \mathbb{R}$. Prove that $W = \{f \in V \mid f(1) = 0\}$ is a subspace of V.

SOLUTION: To show W is a subspace, show three facts: the zero function is in W and that W is closed under addition and scalar multiplication. The zero function is in W because $\theta(t) = 0$ for all $t \in \mathbb{R}$, so $\theta(1) = 0$. If $f, g \in W$, then $f + g \in W$ because $(f + g)(t) = f(t) + g(t)$ so $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$. If $f \in W$ and $b \in \mathbb{R}$ then $bf \in W$ since $(bf)(1) = b(f(1)) = b(0) = 0$.

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(2) (12 points)
(a) T is also a dependent set.
(b) T could be either independent or dependent.
(c) The spans $\langle T_1 \rangle = \langle T \rangle$ are equal. (w_{m+1} is redundant. Also, $\dim(\langle T \rangle) \leq m$.)
(d) $\dim(\langle S_1 \rangle) = k + 1$ (The set S_1 is independent, a basis for $\langle S_1 \rangle$.)
(e) A is invertible, so L_A is invertible, injective, surjective, bijective, an isomorphism, and has inverse $L_{A^{-1}}$.
(f) Since $m < n$, $r = rank(A) \leq Min(m, n) = m$, so $-m \leq -r$ so $n - m \leq n - r$ says the number of free variables in the solution space is at least $n - m$.

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(3) (15 Points) (a) (3 Points)

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 4 \\ 1 & 2 & 0 & 2 & 5 \\ 1 & 3 & 1 & 0 & 6 \\ 1 & 4 & 0 & 2 & 7 \end{bmatrix} \text{ row reduces to } C = \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ so}$$

$\{[1 \quad 0 \quad 0 \quad 2 \quad 3], [0 \quad 1 \quad 0 \quad 0 \quad 1], [0 \quad 0 \quad 1 \quad -2 \quad 0]\}$ is a basis for $Row(A)$.

(b) (3 Points) A basis for $Ker(L_A) = Nul(A)$ is found by row reducing $[A|0_1^4]$ to $[C|0_1^4]$, interpreting the solutions in $\mathbb{R}^5$, and separating the free variables to get two independent vectors which span it:

$$\left\{ \begin{bmatrix} -2r - 3s \\ -s \\ 2r \\ r \\ s \end{bmatrix} = r \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^5 \;\middle|\; r, s \in \mathbb{R} \right\} \text{ so } \left\{ \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $Ker(L_A)$.

(c) (3 Points) A basis for $Col(A)$ consists of the three pivot columns of A, the columns with leading ones in the RREF C, that is, $\{Col_1(A), Col_2(A), Col_3(A)\}$. Other correct answers can be obtained by linear combinations of those three columns, for example,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(d) (6 Points) Each basis vector in $Ker(L_A)$ gives a dependence relation among the columns of A. The two dependence relations obtained that way are:

$$-2Col_1(A)+2Col_3(A)+Col_4(A)=0_1^4 \qquad \text{and} \qquad -3Col_1(A)-1Col_2(A)+1Col_5(A)=0_1^4$$

so

$$Col_4(A)=2Col_1(A)-2Col_3(A) \qquad \text{and} \qquad Col_5(A)=3Col_1(A)+Col_2(A).$$

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(4) (15 points, 3 pts for each part)

(a) Since $9=\dim(\mathbb{F}^9)=\dim(Ker(L))+\dim(Range(L))$ and $0\le\dim(Range(L))\le 4$, we know $5\le\dim(Ker(L))\le 9$.

(b) For $L:\mathbb{F}_3^2\to\mathbb{F}^9$ we know $6=\dim(\mathbb{F}_3^2)=\dim(Ker(L))+\dim(Range(L))$ and $0\le\dim(Ker(L))\le 6$ so $0\le\dim(Range(L))\le 6$.

(c) The set $L(S)=\{L(v_1),\cdots,L(v_n)\}$ spans $Range(L)$. It will only be a basis for $Range(L)$ when L is injective.

(d) $\{[w_1]_T,\cdots,[w_m]_T\}=\{\mathbf{e}_1,\cdots,\mathbf{e}_m\}$ is the standard basis of $\mathbb{F}^m$.

(e) If $A\in\mathbb{F}_n^n$ and $AX=0$ has only the trivial solution, then $Rank(A)=n$.

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(5) (15 Points) Let $L:\mathbb{R}_2^2\to\mathbb{R}^2$ be the linear map given by $L\begin{bmatrix} a & b\\ c & d\end{bmatrix}=\begin{bmatrix} a+2b-c-2d\\ -a+b+c-d\end{bmatrix}$ and let S and T be the standard bases of $\mathbb{R}_2^2$ and $\mathbb{R}^2$, respectively. Let other ordered bases be $S'=\left\{\begin{bmatrix}1&1\\1&1\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ and $T'=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

(a) (2 points) The matrix ${}_T[L]_S=\begin{bmatrix}1&2&-1&-2\\-1&1&1&-1\end{bmatrix}$ by row reducing $[T|L(S)]$.

(b) (3 points) To find $Ker(L)$ row reduce $\begin{bmatrix}1&2&-1&-2&0\\-1&1&1&-1&0\end{bmatrix}$ to $\begin{bmatrix}1&0&-1&0&0\\0&1&0&-1&0\end{bmatrix}$ giving solutions $a=c$, $b=d$ with c and d free variables. Thus, $Ker(L)=\left\{\begin{bmatrix}c&d\\c&d\end{bmatrix}\in\mathbb{R}_2^2 \mid c,d\in\mathbb{R}\right\}$, which has basis $\left\{\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix}\right\}$.

(c) (3 points) ${}_{T'}[L]_{S'}=\begin{bmatrix}0&-3&-6&-3\\0&2&3&1\end{bmatrix}$ since we row reduce

$$\left[\begin{array}{cc|cccc}1&1&0&-1&-3&-2\\1&2&0&1&0&-1\\ \multicolumn{2}{c|}{T'} & \multicolumn{4}{c}{L(S')}\end{array}\right] \quad \text{to} \quad \left[\begin{array}{cc|cccc}1&0&0&-3&-6&-3\\0&1&0&2&3&1\\ \multicolumn{2}{c|}{I_2} & \multicolumn{4}{c}{{}_{T'}[L]_{S'}}\end{array}\right]$$

(d) (4 points) The transition matrices ${}_SP_{S'}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$ and ${}_TQ_{T'}=\begin{bmatrix}1&1\\1&2\end{bmatrix}$ since S and T are the standard bases. Then ${}_{T'}Q_T=({}_TQ_{T'})^{-1}=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}$.

(e) (3 points) The matrix product

$$ {}_{T'}Q_T \ {}_T[L]_S \ {}_SP_{S'} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & -2 \\ -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 3 & 3 & -3 & -3 \\ -2 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & -6 & -3 \\ 0 & 2 & 3 & 1 \end{bmatrix} $$

equals the answer to part (c) as it should.

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(6) (15 Points) Let $L : \mathbb{F}_n^n \to \mathbb{F}_n^n$ be defined by $L(A) = A + A^T$, where A^T means the transpose of A.

(a) (6 pts) Prove that L is a linear map.

Solution: For any $A, B \in \mathbb{F}_n^n$ and any $r \in \mathbb{F}$ we have

$$ L(A+B) = (A+B) + (A+B)^T = A + B + A^T + B^T = A + A^T + B + B^T = L(A) + L(B) $$

and

$$ L(rA) = (rA) + (rA)^T = rA + rA^T = r(A + A^T) = rL(A) $$

so L is linear.

(b) (4 pts) Find $Ker(L)$.

Solution: $Ker(L) = \{A \in \mathbb{F}_n^n \mid A + A^T = 0_n^n\} = \{A \in \mathbb{F}_n^n \mid A^T = -A\}$ is the set of all anti-symmetric matrices in $\mathbb{F}_n^n$.

(c) (5 pts) Use the results in (a) and (b) prove that the set of all anti-symmetric matrices in $\mathbb{F}_n^n$ is a subspace.

Solution: The results in (a) and (b) prove that the set of all anti-symmetric matrices in $\mathbb{F}_n^n$ equals $Ker(L)$, and any kernel of a linear map is a subspace of the domain.

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(7) (15 Points) Let $A \in \mathbb{F}_n^m$ be an $m \times n$ matrix of numbers from field $\mathbb{F}$ with $Rank(A) = r$. Let $\mathbf{0} \in \mathbb{F}^m$ be the $m \times 1$ zero matrix, and let $L_A : \mathbb{F}^n \to \mathbb{F}^m$ be the function associated with A defined as $L_A(X) = AX$. For each part below give the most accurate answer. **No justifications needed**.

(a) If $m > n$ then the function L_A **cannot be surjective**.

(b) $\dim(Ker(L_A)) = n - r$.

(c) $\dim(Range(L_A)) = r$.

(d) Condition $r = n$ is equivalent to L_A being **injective**.

(e) Condition $r = m$ is equivalent to L_A being **surjective**.
