Math 404 Advanced Linear Algebra Spring 2023 Exam 2 Feingold

SHOW WORK IN ORDER TO GET CREDIT FOR YOUR ANSWERS

(1) (15 points, 3 points each) Answer each of the following questions separately.

(a) Let $L : V \to V$ be **invertible**, let $\theta \neq v \in V$ be an eigenvector for L with eigenvalue $\lambda \in \mathbb{F}$. Show that $\lambda \neq 0$ and that v is an eigenvector for L^{-1} with eigenvalue λ^{-1}.

(b) Let $A, B, C \in \mathbb{R}^n_n$ with $det(A) = 2$, $det(B) = -5$ and $\det(C) = 2$. What is the **value** of $det(A^T B^{-1} C^3)$? (A^T means A transpose. Simplify your answer.)

(c) If $A \in \mathbb{F}^n_n$ and the homogeneous linear system $[A|0]$ has only the **trivial solution**, then what is the **most** you can say about $\det(A)$?

(d) Let $L : V \to V$ have r **distinct** eigenvalues $\lambda_1, \cdots, \lambda_r$, with corresponding **algebraic multiplicities** k_i and **geometric multiplicities** g_i for $1 \leq i \leq r$. What relationship between k_i and g_i is **always** true?

(e) Let U and W be subspaces of V with $\dim(V) = 12$, $\dim(U) = 9$ and $\dim(W) = 7$. Find all **possible** values of $\dim(U \cap W)$.

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(2) (10 Points) For $A = \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix}$ find the **characteristic polynomial**, $Char_A(t) = \det(A - tI_4)$, all **eigenvalues** of A and their **algebraic multiplicities**.

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(3) (15 points) The matrix $A = \begin{bmatrix} 4 & 1 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ has **characteristic polynomial** $Char_A(t) = (t-4)^3(t-5)$ so its eigenvalues are $\lambda_1 = 4$ with algebraic multiplicity $k_1 = 3$, and $\lambda_2 = 5$ with algebraic multiplicity $k_2 = 1$.

(a) Can A can be diagonalized? **If not, give reasons why**. If it can, find an **invertible matrix** P and a **diagonal matrix** D such that $D = P^{-1}AP$.

(b) Find the **minimal polynomial** $m_A(t)$ and **justify your answer**.

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(4) (15 Points) Suppose that $A \in \mathbb{R}^7_7$ has **characteristic** and **minimal** polynomials $Char_A(t) = (t-5)^7$ and $m_A(t) = (t-5)^4$. Find all possible **Jordan canonical form** matrices J to which A might be similar, but not to each other, and **for each one give the geometric multiplicity** of the eigenvalue $\lambda_1 = 5$. Use the notation $J(\lambda, n)$ for a basic Jordan block of size $n \times n$ with λ on the diagonal.

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(5) (15 Points) Suppose $A \in \mathbb{R}^{10}_{10}$ has **characteristic** and **minimal** polynomials $Char_A(t) = (t^2+3)^2(t^2+2t+3)^3$ and $m_A(t) = (t^2+3)^2(t^2+2t+3)$. Find all the possible **Rational Canonical Form** matrices that could be similar to A, but not similar to each other. Write out each companion matrix you use in your answer.

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(6) (15 Points) Answer each of the following questions separately.

(a) (5 Points) Find det $\begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 4 & -2 & -3 \end{bmatrix}$.

(b) (3 Points)Let $A = [a_{ij}] \in \mathbb{R}^7_7$ where $a_{ij} = i - j$. Find $\det(A)$.

(c) (2 Points) Suppose $\dim(V) = n$, $L : V \to V$ has r **distinct** eigenvalues $\lambda_1, \cdots, \lambda_r$, and for $1 \leq i \leq r$ let $T_i = \{v_{ij} \mid 1 \leq j \leq g_i\}$ be a **basis** of the λ_i eigenspace, L_{λ_i}. What **relation** between n and the **geometric multiplicities** g_i means that L is **diagonalizable**?

(d) (5 Points) Let $V = \mathbb{F}^n_n$, let $U = \{A \in V \mid A^T = A\}$ be the subspace of symmetric matrices, and let $W = \{A \in V \mid A^T = -A\}$ be the subspace of anti-symmetric matrices. Prove that $V = U + W$ and that the sum is a **direct sum**.

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(7) (15 Points) Let $L : \mathbb{R}^4 \to \mathbb{R}^4$ be $L(X) = AX$ for $A = \begin{bmatrix} 0 & 12 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ 0 & -9 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$.

Let $S = \{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$ and let
$T = \{v_1 = e_1, v_2 = L(e_1), v_3 = L^2(e_1), v_4 = L^3(e_1)\}$.

(a) (5 pts) Find T and show it is independent, so it is a basis of $\mathbb{R}^4$.

(b) (4 pts) Find $L(v_4) = L^4(e_1)$ and write it as a linear combination of the basis vectors in T.

(c) (3 pts) Using the answers to parts (a) and (b) find the **companion matrix** $C = {}_T[L]_T$ that represents L with respect to T.

(d) (3 pts) Using your answer to part (c) give the **characteristic polynomial**, $Char_L(t)$, and the **minimal polynomial**, $m_L(t)$.

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(1) (15 points, 3 points each)

(a) If $\lambda = 0$ then $L(v) = \lambda v = \theta$ would mean $v \in Ker(L)$ is nontrivial, so L would not be injective, contradicting that L is invertible. From $L(v) = \lambda v$ we get $v = L^{-1}(L(v)) = L^{-1}(\lambda v) = \lambda L^{-1}(v)$ so $\lambda^{-1} v = L^{-1}(v)$ which makes v an eigenvector for L^{-1} with eigenvalue λ^{-1}.

(b) $det(A^T B^{-1} C^3) = \dfrac{det(A)det(C)^3}{det(B)} = \dfrac{(2)(2^3)}{-5} = \dfrac{16}{-5}$.

(c) If $[A|0]$ has only the trivial solution then A row reduces to the identity matrix, so A is invertible, so $\det(A) \neq 0$.

(d) The relationship $g_i \leq k_i$ for $1 \leq i \leq r$ is always true.

(e) Since $\dim(V) = 12$, $\dim(U) = 9$ and $\dim(W) = 7$, and
$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W) = 9 + 7 - \dim(U \cap W)$, and
$\dim(U + W) \leq \dim(V) = 12$, we have $16 - \dim(U \cap W) \leq \dim(V) = 12$ so
$4 \leq \dim(U \cap W)$. Also, $\dim(U \cap W) \leq Min(\dim(U), \dim(W) = 7$, so
$4 \leq \dim(U \cap W) \leq 7$.

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(2) (10 points) The characteristic polynomial is $Char_A(t) = \det(tI_4 - A) = \det(A - tI_4) =$

$$\det \begin{bmatrix} 18-t & -20 & -20 & -20 \\ 5 & -7-t & -5 & -5 \\ 5 & -5 & -7-t & -5 \\ 5 & -5 & -5 & -7-t \end{bmatrix} = \det \begin{bmatrix} -t-2 & 0 & 0 & 4t+8 \\ 0 & -t-2 & 0 & t+2 \\ 0 & 0 & -t-2 & t+2 \\ 5 & -5 & -5 & -7-t \end{bmatrix}$$

$$= (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 5 & -5 & -5 & -7-t \end{bmatrix} = (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -5 & -5 & -t+13 \end{bmatrix}$$

$$= (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -5 & -t+8 \end{bmatrix} = (t+2)^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -t+3 \end{bmatrix}$$

$$= (t+2)^3 \ (t-3).$$

So the eigenvalues are $\lambda_1 = -2$ with algebraic multiplicity $k_1 = 3$ and $\lambda_2 = 3$ with algebraic multiplicity $k_2 = 1$.

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(3) (15 points) The matrix $A = \begin{bmatrix} 4 & 1 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ has **characteristic polynomial** $Char_A(t) = (t-4)^3(t-5)$ so its eigenvalues are $\lambda_1 = 4$ with algebraic multiplicity $k_1 = 3$, and $\lambda_2 = 5$ with algebraic multiplicity $k_2 = 1$. (a) Can A can be diagonalized? (b) Find the **minimal polynomial** $m_A(t)$ and **justify your answer**.

Solution: (a) Check the $\lambda_1 = 4$ eigenspace A_{λ_1} first since the power of $t-4$ in $Char_A(t)$ is $k_1 = 3$. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = 4$ to $A - tI_4$. Row reduce

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r \in \mathbb{R} \\ x_2 = 0 \\ x_3 = s \in \mathbb{R} \\ x_4 = 0 \end{array}$$

so $g_1 = \dim(A_{\lambda_1}) = 2 < 3 = k_1$ means A is **not diagonalizable**.

(b) The minimal polynomial $m_A(t)$ divides $Char_A(t)$ and has the same linear factors, so it must be of the form $m_A(t) = (t-4)^m(t-5)$ for $1 \le m \le 3$. We compute $(A-4I_4)(A-5I_4) =$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0_4^4 \text{ but}$$

$$(A-4I_4)^2(A-5I_4) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0_4^4 \text{ so } m_A(t) = (t-4)^2(t-5).$$

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(4) (15 Points) Suppose that $A \in \mathbb{R}_7^7$ has **characteristic** and **minimal** polynomials $Char_A(t) = (t-5)^7$ and $m_A(t) = (t-5)^4$. Find all possible **Jordan canonical form** matrices J to which A might be similar, but not to each other, and **for each one give the geometric multiplicity** of the eigenvalue $\lambda_1 = 5$.

SOLUTION: $Char_A(t) = (t-5)^7$ and $m_A(t) = (t-5)^4$ so there is only one eigenvalue, $\lambda_1 = 5$ with algebraic multiplicity $k_1 = 7$. The power in the minimal polynomial $m_1 = 4$ is the size of the largest Jordan block. Let

$$B = J(5,4) = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \; C = J(5,3) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } D = J(5,2) = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}.$$

Then there are 3 possible Jordan canonical form matrices similar to A, corresponding to the partitions of 7 into parts with largest part 4:

Partition :	$4+3$	$4+2+1$	$4+1+1+1$
Jordan Form :	$Diag(B,C)$	$Diag(B,D,5)$	$Diag(B,5,5,5)$
Geom. Mult. (number of J − blocks) :	2	3	4

(5) (15 Points) Suppose $A \in \mathbb{R}^{10}_{10}$ has **characteristic** and **minimal** polynomials $Char_A(t) = (t^2+3)^2(t^2+2t+3)^3$ and $m_A(t) = (t^2+3)^2(t^2+2t+3)$. Find all the possible **Rational Canonical Form** matrices that could be similar to A, but not similar to each other. Write out each companion matrix you use in your answer.

SOLUTION: Since there are two irreducible factors in $Char_A(t)$, there are two kinds of blocks in the RCF similar to A. The blocks coming from $(t^2+3)^2$ have a total size of 4, the degree of this factor, and the blocks coming from $(t^2+2t+3)^3$ have a total size of 6 for the same reason. But because in $m_A(t)$ we have $(t^2+3)^2$, there must be only one 4×4 companion matrix $C = \begin{bmatrix} 0 & 0 & 0 & -9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \end{bmatrix} = C((t^2+3)^2) = C(t^4+6t^2+9)$.

Since in $m_A(t)$ we have (t^2+2t+3), there must be three 2×2 companion matrices $D = \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix} = C(t^2+2t+3)$. So the only possible RCF is $diag(C, D, D, D)$.

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(6) (15 Points) Answer each of the following questions separately.

(a) (5 Points)

$$\det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 4 & -2 & -3 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & -1 & 4 & 4 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 10 & 7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} = -\det\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -4$$

(b) (3 Points) Since $A = [a_{ij}] \in \mathbb{R}^7_7$ where $a_{ij} = i - j$, we see that $a_{ji} = j - i = -a_{ij}$ so $A^T = -A$ is skew-symmetric. Then $\det(A) = \det(A^T) = \det(-A) = (-1)^7\det(A) = -\det(A)$, so $2\det(A) = 0$ so $\det(A) = 0$.

(c) (2 Points) Suppose $\dim(V) = n$, $L : V \to V$ has r **distinct** eigenvalues $\lambda_1, \cdots, \lambda_r$, and for $1 \le i \le r$ let $T_i = \{v_{ij} \mid 1 \le j \le g_i\}$ be a **basis** of the λ_i eigenspace, L_{λ_i}. What relation between n and the **geometric multiplicities** g_i means that L is diagonalizable?

Solution: L is diagonalizable when $g_1 + \cdots + g_r = n$.

(d) (5 Points) Let $V = \mathbb{F}^n_n$, let $U = \{A \in V \mid A^T = A\}$ be the subspace of symmetric matrices, and let $W = \{A \in V \mid A^T = -A\}$ be the subspace of anti-symmetric matrices. Prove that $V = U + W$ and that the sum is a **direct sum**.

Solution: For any $A \in V$ let $A_{sym} = (A + A^T)/2$ and let $A_{anti} = (A - A^T)/2$. Then $A^T_{sym} = A_{sym}$ and $A^T_{anti} = -A_{anti}$ so $A_{sym} \in U$ and $A_{anti} \in W$. So $A = A_{sym} + A_{anti} \in U + W$ proves that $V = U + W$. The sum is direct because if $A \in U \cap W$ then $A = A^T = -A$ implies $A = 0^n_n$.

(7) (15 Points) Let $L : \mathbb{R}^4 \to \mathbb{R}^4$ be $L(X) = AX$ for $A = \begin{bmatrix} 0 & 12 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ 0 & -9 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$.

Let $S = \{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$ and let $T = \{v_1 = e_1, v_2 = L(e_1), v_3 = L^2(e_1), v_4 = L^3(e_1)\}$.

(a) (5 pts) Find T and show it is independent, so it is a basis of $\mathbb{R}^4$.

Solution: $T = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 12 \\ -4 \\ -9 \\ 2 \end{bmatrix} \right\}$. To show it is independent, show $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = \theta$ has only the trivial solution. We row reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 12 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & -9 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{array}$$

(b) (4 pts) Find $L(v_4) = L^4(e_1)$ and write it as a linear combination of the basis vectors in T.

Solution: $L(v_4) = L^4(e_1) = \begin{bmatrix} -57 \\ 18 \\ 36 \\ 4 \end{bmatrix}$. To solve $\sum_{i=1}^4 x_iv_i = L(v_4)$ row reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 12 & -57 \\ 0 & 0 & 1 & -4 & 18 \\ 0 & 0 & 0 & -9 & 36 \\ 0 & 1 & 0 & 2 & 4 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -9 \\ x_2 = 12 \\ x_3 = 2 \\ x_4 = -4 \end{array}$$

This means $L(v_4) = -9v_1 + 12v_2 + 2v_3 - 4v_4$.

(c) (3 pts) Using the answers to parts (a) and (b) find the **companion matrix** $C = {}_T[L]_T$ that represents L with respect to T.

Solution: From the answers to parts (a) and (b), the **companion matrix** that represents L from T to T is

$$C = {}_T[L]_T = \begin{bmatrix} 0 & 0 & 0 & -9 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

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(d) (3 pts) Using your answer to part (c) give the **characteristic polynomial**, $Char_L(t)$, and the **minimal polynomial**, $m_L(t)$.

Solution: From the answer to part (c) the **characteristic polynomial** equals the **minimal polynomial**,

$$Char_L(t) = m_L(t) = t^4 + 4t^3 - 2t^2 - 12t + 9.$$