Math 404 Advanced Linear Algebra Spring 2023 Exam 3 Feingold

SHOW ALL NECESSARY WORK FOR EACH PROBLEM

(1) (20 Points, 4 points each) Answer each part separately.

(a) For $X = [x_i], Y = [y_i] \in \mathbb{R}^n$ with the **standard dot product**, write out what the Cauchy-Schwarz inequality says about all the numbers x_i and y_i, $1 \leq i \leq n$.

(b) Suppose $A \in \mathbb{R}^n_n$ is a matrix whose columns $C_j = Col_j(A)$, form an **orthogonal** set of non-zero vectors in $\mathbb{R}^n$. What is the **most** you can say about $A^{Tr}A$?

(c) Let $S = \{v_1, \cdots, v_n\}$ be an **orthogonal basis** of inner product space V, so for any $v \in V$ we can write $v = \sum_{i=1}^n c_i v_i$. Use the inner product $\langle \cdot, \cdot \rangle$ to give a **formula** for the coefficients c_i.

(d) Let $v = [\,1 \quad 2 \quad 3\,] \in \mathbb{R}_3$ with the standard dot product. Find a **basis** for the subspace $v^\perp = \{X \in \mathbb{R}_3 \mid X \cdot v = 0\}$.

(e) Suppose V is a real inner product space with inner product $\langle \cdot, \cdot \rangle$. Let $T = \{v_1, \cdots, v_m\}$ be an **orthogonal basis** for a subspace $W \leq V$. Write the formula for the **projection** $Proj_W(v)$ of any vector $v \in V$ onto W.

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(2) (20 Points) Let $V = P_2(\mathbb{R}) = \{a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R}\}$ be the inner product space with inner product $\langle p, q \rangle = \int_0^1 p(t)q(t)dt$. Let $S = \{1, t, t^2\}$ be the standard basis of V.

(a) (10 points) For $i, j \in \{0, 1, 2\}$ compute $\langle t^i, t^j \rangle$ and use the results to get the 3×3 matrix $M = [\langle t^i, t^j \rangle]$ representing the inner product with respect to S.

(b) (10 points) Use the Gram-Schmidt process to orthogonalize S but **do not** do the normalizing step.

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(3) (20 Points) Let $V = \mathbb{R}^4$ with the standard dot product, and let $W = S^\perp$, the orthogonal complement in V of $S = \{u_1 = [\,1 \quad 1 \quad 1 \quad 1\,]^{Tr}, u_2 = [\,1 \quad 1 \quad 2 \quad 3\,]^{Tr}\}$.

(a) (5 pts) Find a basis $T = \{w_1, w_2\}$ for W.

(b) (10 pts) Use Gram-Schmidt to get a nice **orthogonal** basis $T' = \{w'_1, w'_2\}$ for W.

(c) (5 pts) Use T' to find the **coefficients** x_i of the projection $Proj_W(v) = x_1 w'_1 + x_2 w'_2$ of the general vector $v = [\,a \quad b \quad c \quad d\,]^{Tr} \in V$ into W.

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(4) (20 points, 4 points each) Answer each part separately. Let $A^* = \overline{A}^{Tr}$.

(a) Suppose $A = A^{Tr} \in \mathbb{R}^n_n$ and $\lambda \neq \mu$ are two distinct eigenvalues of A with corresponding eigenvectors, $X, Y \in \mathbb{R}^n$, so $AX = \lambda X$ and $AY = \mu Y$. Prove that $X \cdot Y = 0$.

(b) What equation must $A \in \mathbb{R}^n_n$ satisfy in order for A to be called an **orthogonal** matrix?

(c) What equation must $A \in \mathbb{C}^n_n$ satisfy in order for A to be called a **unitary** matrix?

(d) What conditions must $M = M^{Tr} \in \mathbb{R}^n_n$ satisfy for M to be called **positive definite**?

(e) Suppose $A = A^* \in \mathbb{C}^n_n$ is Hermitian and $\lambda \in \mathbb{C}$ is an eigenvalue of A with eigenvector $X \in \mathbb{C}^n$, so $AX = \lambda X$. Prove that $\lambda \in \mathbb{R}$ is real.

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(5) (20 Points) Let V be a real vector space with basis $S = \{v_1, \cdots, v_n\}$. A bilinear form $\langle \cdot, \cdot \rangle$ on V is determined by ${}_S M_S = [\langle v_i, v_j \rangle] \in \mathbb{R}^n_n$ since for any $u, v \in V$,

$$\langle u, v \rangle = [u]_S^{Tr} \, {}_S M_S [v]_S.$$

Answer the following questions about this bilinear form.

(a) What conditions on $M = {}_S M_S$ mean that the bilinear form is an **inner product**?

(b) Let $T = \{w_1, \cdots, w_n\}$ be another basis of V and let ${}_T M_T = [\langle w_i, w_j \rangle]$ so we also have

$$\langle u, v \rangle = [u]_T^{Tr} \, {}_T M_T [v]_T.$$

Let $P = {}_S P_T$ be the transition matrix from T to S. What is the equation giving the relationship between the matrices $M = {}_S M_S$, ${}_T M_T$ and P? **Justify your answer**.

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(1) (20 Points, 4 points each) Answer each part separately.

(a) For $X = [x_i], Y = [y_i] \in \mathbb{R}^n$ with the standard dot product, write out what the Cauchy-Schwarz inequality says about all the numbers x_i and y_i, $1 \leq i \leq n$.

Solution: For any $X = [x_i], Y = [y_i] \in \mathbb{R}^n$, the Cauchy-Schwarz inequality says

$(X \cdot Y)^2 \leq (X \cdot X)\ (Y \cdot Y)$, which means $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\ \left(\sum_{i=1}^{n} y_i^2\right)$.

(b) Suppose $A \in \mathbb{R}^n_n$ is a matrix whose columns $C_j = Col_j(A)$, form an **orthogonal** set of non-zero vectors in $\mathbb{R}^n$. What is the **most** you can say about $A^{Tr}A$?

Solution: Since the (i,j)-entry of $A^{Tr}A$ equals $C_i^{Tr}C_j = C_i \cdot C_j$ we know that is 0 when $i \neq j$, so the matrix $A^{Tr}A$ must be **diagonal**, with the numbers $C_i \cdot C_i > 0$ on the diagonal.

(c) Let $S = \{v_1, \cdots, v_n\}$ be an **orthogonal basis** of inner product space V, so for any $v \in V$ we can write $v = \sum_{i=1}^{n} c_i v_i$. Use the inner product $\langle \cdot, \cdot \rangle$ to give a formula for the coefficients c_i.

Solution: For each $1 \leq j \leq n$ we have

$$\langle v, v_j \rangle = \sum_{i=1}^{n} c_i \langle v_i, v_j \rangle = \sum_{i=1}^{n} c_i \delta_{ij} \langle v_i, v_j \rangle = c_j \langle v_j, v_j \rangle \quad \text{so} \quad c_j = \frac{\langle v, v_j \rangle}{\langle v_j, v_j \rangle}.$$

(d) Let $v = [\,1 \quad 2 \quad 3\,] \in \mathbb{R}_3$ with the standard dot product. Find a basis for the subspace $v^\perp$.

Solution: By definition, $v^\perp = \{X = [\,x_1 \quad x_2 \quad x_3\,] \in \mathbb{R}_3 \mid X \cdot v = 0\}$. But $X \cdot v = 1x_1 + 2x_2 + 3x_3$ and the single linear equation $1x_1 + 2x_2 + 3x_3 = 0$ has solutions $x_1 = -2r - 3s$ where $x_2 = r$ and $x_3 = s$ are free variables. So

$$v^\perp = \{[\,(-2r - 3s) \quad r \quad s\,] \in \mathbb{R}_3 \mid r, s \in \mathbb{R}\}.$$

which has basis $\{[\,-2 \quad 1 \quad 0\,], [\,-3 \quad 0 \quad 1\,]\}$.

(e) Suppose V is a real inner product space with inner product $\langle \cdot, \cdot \rangle$. Let $T = \{v_1, \cdots, v_m\}$ be an **orthogonal basis** for a subspace $W \leq V$. Write the formula for the **projection** $Proj_W(v)$ of any vector $v \in V$ onto W.

Solution: The formula for the projection is $Proj_W(v) = \sum_{i=1}^{m} \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} v_i$.

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(2) (20 Points) Let $V = P_2(\mathbb{R}) = \{a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R}\}$ be the inner product space with inner product $\langle p, q\rangle = \int_0^1 p(t)q(t)dt$. Let $S = \{1, t, t^2\}$ be the standard basis of V.

(a) (10 points) For $i, j \in \{0, 1, 2\}$ compute $\langle t^i, t^j\rangle$ and use the results to get the 3×3 matrix $M = [(\langle t^i, t^j\rangle]$ representing the inner product with respect to S.

Solution:

$$\langle t^i, t^j\rangle = \int_0^1 t^{i+j} dt = \left.\frac{t^{i+j+1}}{i+j+1}\right|_0^1 = \frac{1}{i+j+1}$$

so the matrix

$$M = [\langle t^i, t^j\rangle] = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

(b) (10 points) Use the Gram-Schmidt process to orthogonalize S but **do not** do the normalizing step.

Solution: The first step of Gram-Schmidt orthogonalization does not change the first standard basis vector 1, but the second step does change t to

$$t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle} 1 = t - \frac{\frac{1}{2}}{1} 1 = t - \frac{1}{2}.$$

The third step changes t^2 to

$$\begin{aligned}
&t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle t^2, t - \frac{1}{2}\rangle}{\langle t - \frac{1}{2}, t - \frac{1}{2}\rangle}(t - \frac{1}{2}) = \\
&t^2 - \frac{1}{3} - \frac{\frac{1}{4} - \frac{1}{2}\frac{1}{3}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}}(t - \frac{1}{2}) = \\
&t^2 - \frac{1}{3} - \frac{\frac{1}{12}}{\frac{1}{12}}(t - \frac{1}{2}) = \\
&t^2 - \frac{1}{3} - (t - \frac{1}{2}) = \\
&t^2 - t + \frac{1}{6}.
\end{aligned}$$

Then the orthogonal basis of V we obtained by Gram-Schmidt from S is

$$\{1, \;\; t - \frac{1}{2}, \;\; t^2 - t + \frac{1}{6}\}.$$

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(3) (20 Points) Let $V = \mathbb{R}^4$ with the standard dot product, and let $W = S^\perp$, the orthogonal complement in V of $S = \{u_1 = [1 \quad 1 \quad 1 \quad 1]^{Tr}, u_2 = [1 \quad 1 \quad 2 \quad 3]^{Tr}\}$.

(a) (5 pts) Find a basis $T = \{w_1, w_2\}$ for W.

Solution: $W = \{X \in \mathbb{R}^4 \mid X \cdot u_i = 0, \ i = 1, 2\}$ is found by row reducing

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 3 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -r + s \\ x_2 = r \\ x_3 = -2s \\ x_4 = s \in \mathbb{R} \end{array} \text{ so } T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

(b) (10 pts) Use Gram-Schmidt to get a nice **orthogonal** basis $T' = \{w_1', w_2'\}$ for W.

Solution: Gram-Schmidt gives $w_1' = w_1$,

$$w_2' = w_2 - \frac{w_2 \cdot w_1'}{w_1' \cdot w_1'} w_1' = w_2 - \frac{-1}{2} w_1' = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -4 \\ 2 \end{bmatrix} \text{ so } T' = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -4 \\ 2 \end{bmatrix} \right\}$$

rescaling to avoid fractions in the last vector.

(c) (5 pts) Use T' to find the **coefficients** x_i of the projection $Proj_W(v) = x_1 w_1' + x_2 w_2'$ of the general vector $v = [a \quad b \quad c \quad d]^{Tr} \in V$ into W.

Solution: Since T' is an orthogonal basis of W, we have

$$x_i = \frac{v \cdot w_i'}{w_i' \cdot w_i'} \quad \text{so} \quad x_1 = \frac{-a + b}{2}, \quad x_2 = \frac{a + b - 4c + 2d}{22}.$$

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(4) (20 points, 4 points each) Answer each part separately. Let $A^* = \overline{A}^{Tr}$.

(a) Suppose $A = A^{Tr} \in \mathbb{R}^n_n$ and $\lambda \neq \mu$ are two distinct eigenvalues of A with corresponding eigenvectors, $X, Y \in \mathbb{R}^n$, so $AX = \lambda X$ and $AY = \mu Y$. Prove that $X \cdot Y = 0$.

Solution: We have $\lambda(X \cdot Y) = (\lambda X) \cdot Y = (AX) \cdot Y = X \cdot (A^{Tr} Y) = X \cdot (AY) = X \cdot (\mu Y) = \mu(X \cdot Y)$ so $(\lambda - \mu)(X \cdot Y) = 0$ so $X \cdot Y = 0$ since $\lambda - \mu \neq 0$.

(b) What equation must $A \in \mathbb{R}^n_n$ satisfy in order for A to be called an **orthogonal** matrix?

Solution: A must satisfy the equation $A^{Tr} = A^{-1}$ or $A^{Tr} A = I_n$.

(c) What equation must $A \in \mathbb{C}^n_n$ satisfy in order for A to be called a **unitary** matrix?

Solution: A must satisfy the equation $A^* = \overline{A}^{Tr} = A^{-1}$ or $\overline{A}^{Tr} A = I_n$.

(d) What conditions must $M = M^{Tr} \in \mathbb{R}^n_n$ satisfy for M to be called **positive definite**?

Solution: M must satisfy $X^{Tr} M X \geq 0$ for all $X \in \mathbb{R}^n$ and $X^{Tr} M X = 0$ implies $X = 0^n_1$.

(e) Suppose $A = A^* \in \mathbb{C}^n_n$ is Hermitian and $\lambda \in \mathbb{C}$ is an eigenvalue of A with eigenvector $X \in \mathbb{C}^n$, so $AX = \lambda X$. Prove that $\lambda \in \mathbb{R}$ is real.

Solution: We have $\lambda(X \cdot X) = (\lambda X) \cdot X = (AX) \cdot X = X \cdot (A^* X) = X \cdot (AX) = X \cdot (\lambda X) = \bar{\lambda}(X \cdot X)$ so $(\lambda - \bar{\lambda})(X \cdot X) = 0$ so $\lambda - \bar{\lambda} = 0$ since $X \cdot X > 0$. Thus, $\lambda = \bar{\lambda}$ is real.

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(5) (20 Points) Let V be a real vector space with basis $S = \{v_1, \cdots, v_n\}$. A bilinear form $\langle \cdot, \cdot \rangle$ on V is determined by ${}_SM_S = [\langle v_i, v_j \rangle] \in \mathbb{R}^n_n$ since for any $u, v \in V$,

$$\langle u, v \rangle = [u]_S^{Tr}\, {}_SM_S [v]_S.$$

Answer the following questions about this bilinear form.

(a) What conditions on $M = {}_SM_S$ mean that the bilinear form is an **inner product**?

Solution: An inner product must be symmetric and positive definite, so the corresponding conditions on $M = {}_SM_S$ are that the matrix is symmetric and positive definite, that is, $M = M^{Tr}$ and for any $X \in \mathbb{R}^n$, $X^{Tr}MX \geq 0$ and $X^{Tr}MX = 0$ implies $X = 0^n_1$.

(b) Let $T = \{w_1, \cdots, w_n\}$ be another basis of V and let ${}_TM_T = [\langle w_i, w_j \rangle]$ so we also have

$$\langle u, v \rangle = [u]_T^{Tr}\, {}_TM_T [v]_T.$$

Let $P = {}_SP_T$ be the transition matrix from T to S. What is the equation giving the relationship between the matrices $M = {}_SM_S$, ${}_TM_T$ and P? **Justify your answer**.

Solution: The relationship is ${}_TM_T = P^{Tr}MP$ because for any $u, v \in V$, we have

$$\langle u, v \rangle = [u]_S^{Tr}\, {}_SM_S [v]_S = [u]_T^{Tr}\, {}_TM_T [v]_T$$

and ${}_SP_T[u]_T = [u]_S$ and ${}_SP_T[v]_T = [v]_S$ so

$$\langle u, v \rangle = (P[u]_T)^{Tr} MP[v]_T = [u]_T^{Tr}\, {}_TM_T [v]_T$$

which gives that for any $u, v \in V$, we have

$$[u]_T^{Tr}(P^{Tr}MP)[v]_T = [u]_T^{Tr}\, {}_TM_T [v]_T.$$

Using $u = w_i \in T$ and $v = w_j \in T$ we have $[w_i]_T = \mathbf{e}_i$ and $[w_j]_T = \mathbf{e}_j$ standard basis vectors in $\mathbb{R}^n$, so

$$\mathbf{e}_i^{Tr}(P^{Tr}MP)\mathbf{e}_j = \mathbf{e}_i^{Tr}\, {}_TM_T \mathbf{e}_j$$

which says the $(i, j)^{th}$ entries of $P^{Tr}MP$ and ${}_TM_T$ are equal, so $P^{Tr}MP = {}_TM_T$.
