NAME (Printed): ______________________________

Math 404 Advanced Linear Algebra Spring 2023 Quiz 10 Feingold

Show all work.

(1) (2 Pts) Show that $M = [m_{ij}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is **positive definite**.

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(2) (3 Pts) The matrix M from (1) defines an inner product on $\mathbb{R}^3$ by the formula $\langle X, Y\rangle = X^T M Y$. Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$ and let $\theta_{X,Y}$ be the angle between X and Y in the geometry determined by M. Then:

$\cos(\theta_{\mathbf{e}_1,\mathbf{e}_2}) =$ $\qquad$ $\cos(\theta_{\mathbf{e}_2,\mathbf{e}_3}) =$ $\qquad$ and $\qquad$ $\cos(\theta_{\mathbf{e}_1,\mathbf{e}_3}) =$

(1) (2 Pts) Show that $M = [m_{ij}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is **positive definite**.

Solution: We have $[\,x \quad y \quad z\,] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz =$ $x^2 + (x-y)^2 + (y-z)^2 + z^2 \geq 0$ since it is the sum of real squares. That expression is 0 iff $0 = x = x - y = y - z = z$ iff $x = y = z = 0$ so M is positive definite.

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Solution: Let $M = [m_{ij}]$ from (1). Since $\langle \mathbf{e}_i, \mathbf{e}_j\rangle = \mathbf{e}_i^T M \mathbf{e}_j = m_{ij}$ and

$$\cos(\theta_{\mathbf{e}_i,\mathbf{e}_j}) = \frac{\langle \mathbf{e}_i, \mathbf{e}_j\rangle}{(||\mathbf{e}_i||)(||\mathbf{e}_j||)} = \frac{m_{ij}}{\sqrt{2}\sqrt{2}} = \frac{m_{ij}}{2}$$

we get

$$\cos(\theta_{\mathbf{e}_1,\mathbf{e}_2}) = \frac{-1}{2} \qquad \cos(\theta_{\mathbf{e}_2,\mathbf{e}_3}) = \frac{-1}{2} \qquad \text{and} \qquad \cos(\theta_{\mathbf{e}_1,\mathbf{e}_3}) = 0$$
