NAME (Printed): ____

Math 404 Advanced Linear Algebra Spring 2023 Quiz 5 Feingold

Show all calculations and reasons needed to justify your answers.

The real matrix $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ has characteristic polynomial $t^2\ (t-2)\ (t+2)$.

(1) (3 Points) For each eigenvalue of A find a **basis** of the corresponding **eigenspace**.

(2) (2 Points) Find a diagonal matrix D and a transition matrix P such that $D = P^{-1}AP$.

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(1) (3 Points) Since $\det(A - tI_4) = t^2\ (t-2)\ (t+2)$, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = -2$.

The $\lambda_1 = 0$ eigenspace is found by row reducing $[A - 0I_4|0_1^4] =$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -r \\ x_2 = -s \\ x_3 = r \in \mathbb{R} \\ x_4 = s \in \mathbb{R} \end{array}, \text{ so}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} -r \\ -s \\ r \\ s \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; r, s \in \mathbb{R} \right\} \text{ has basis } \left\{ w_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The $\lambda_2 = 2$ eigenspace is found by row reducing $[A - 2I_4|0_1^4] =$

$$\left[\begin{array}{cccc|c} -2 & 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -2 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r \\ x_2 = r \\ x_3 = r \\ x_4 = r \in \mathbb{R} \end{array}, \text{ so}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; r \in \mathbb{R} \right\} \text{ has basis } \left\{ w_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

The $\lambda_3 = -2$ eigenspace is found by row reducing $[A + 2I_4|0_1^4] =$

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -r \\ x_2 = r \\ x_3 = -r \\ x_4 = r \in \mathbb{R} \end{array}, \text{ so}$$

$$A_{\lambda_3} = \left\{ \begin{bmatrix} -r \\ r \\ -r \\ r \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; r \in \mathbb{R} \right\} \text{ has basis } \left\{ w_4 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

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(2) (2 Points) Since $T = \{w_1, w_2, w_3, w_4\}$ is independent, it is an e-basis for $\mathbb{R}^4$ and A is similar to the diagonal matrix $D = P^{-1}AP$ where

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = {}_S P_T$$

is the transition matrix from T to the standard basis S of $\mathbb{R}^4$, so $Col_j(P) = w_j$ for $1 \leq j \leq 4$.
