NAME (Printed): ____

Math 404 Advanced Linear Algebra Spring 2025 Quiz 7 Feingold

Show all calculations and reasons needed to justify your answers.

(1) (5 Points) Suppose that $A \in \mathbb{R}^{10}_{10}$ has **characteristic** and **minimal** polynomials

$$char_A(t) = (t^2+2)^3(t^2-t+1)^2 \quad \text{and} \quad m_A(t) = (t^2+2)^2(t^2-t+1) \ .$$

Find all **Rational Canonical Form** matrices R similar to A but not to each other.

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Solution: The **characteristic and minimal polynomials** are $char_A(t) = (f_1(t))^3(f_2(t))^2$ and $m_A(t) = (f_1(t))^2 f_2(t)$ where the factors $f_1(t) = t^2+2$ and $f_2(t) = t^2-t+1$ are irreducible over $\mathbb{R}$. Define the companion matrices $C_1 = C((t^2+2)^2) = C(t^4+4t^2+4)$, $C_2 = C(t^2+2)$ and $C_3 = C(t^2-t+1)$, so

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \qquad \text{and} \qquad C_3 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since the exponents in $m_A(t)$ give the sizes of the largest companion blocks for each irreducible factor, **there is only one possible RCF matrix similar to A**,

$$R = Diag(C_1, C_2, C_3, C_3).$$

This is the only way to get the given $char_A(t)$ as the **product** of the characteristic polynomials of each companion block, and the given minimal polynomial $m_A(t)$ as the **least common multiple** of the minimal polynomials of those companion blocks. Putting these in any different order only gives a matrix similar to this one.
