NAME (Printed): ____

Math 404 Advanced Linear Algebra Spring 2025 Quiz 8 Feingold

Show all calculations and reasons needed to justify your answers.

Let $V = \mathbf{R}^5$ with the standard dot product. Let $W = \langle T \rangle$ where

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, w_3 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is an ordered basis of } W.$$

(1) (5 Pts) Use the Gram-Schmidt process to convert T into an **orthogonal** basis $T' = \{w_1' = w_1, w_2', w_3'\}$ for W. **Rescale w_3' to avoid fractions in your final answer**.

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Solution: Step 1: $w_1' = w_1$.

Step 2: $$w_2' = w_2 - \left(\frac{w_2 \cdot w_1'}{w_1' \cdot w_1'} \right) w_1' = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \frac{15}{5} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Step 3: $$w_3' = w_3 - \left(\frac{w_3 \cdot w_1'}{w_1' \cdot w_1'} \right) w_1' - \left(\frac{w_3 \cdot w_2'}{w_2' \cdot w_2'} \right) w_2' = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{5} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{10} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10} \left(\begin{bmatrix} -10 \\ -10 \\ 0 \\ 0 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \\ 0 \\ -5 \\ -10 \end{bmatrix} \right) = \frac{1}{10} \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \\ 2 \end{bmatrix}.$$

So, rescaling w_3' to avoid fractions, $$T' = \left\{ w_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2' = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, w_3' = \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \\ 2 \end{bmatrix} \right\}.$$

By the process, $\langle T' \rangle = \langle T \rangle = W$, and we check that $w_i' \cdot w_j' = 0$ for $1 \leq i < j \leq 3$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \\ 2 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \\ 2 \end{bmatrix} = 0.$$