Math 507 Linear Algebra and Matrix Theory Fall 2019 Practice Exam 2 Feingold

1. Answer each question separately. Let A^T mean the transpose of A.

   (a) Let $W_1, W_2 \leq V$ be subspaces with $\dim(V) = 10$, $\dim(W_1) = 6$ and $\dim(W_2) = 7$. Find all possibilities for $\dim(W_1 \cap W_2)$.
   (b) What is $\dim(Hom(\mathbf{F}_3^2, \mathbf{F}_4^5))$? What isomorphism justifies this?
   (c) If $A^T = -A$ for a matrix $A \in \mathbf{R}_n^n$ where n is odd, what is the **most** you can say about $det(A)$?
   (d) Let $A, B, C \in \mathbf{R}_n^n$ with $det(A) = 3$, $det(B) = -7$ and $\det(C) = 2$. Find $det(A^T B^{-1} C^3)$.
   (e) Find $\det \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix}$.

2. Let $\langle \cdot, \cdot \rangle : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be defined by $\langle X, Y \rangle = X^T M Y$ for a matrix $M \in \mathbf{R}_n^n$.

   (a) Prove that $\langle X, Y \rangle$ is **bilinear**.
   (b) Prove that $\langle X, Y \rangle = \langle Y, X \rangle$ iff $M = M^T$ is symmetric.
   (c) Find the condition on $A \in \mathbf{R}_n^n$ which means $\langle X, Y \rangle = \langle AX, AY \rangle$ for all $X, Y \in \mathbf{R}^n$.

3. The set $S = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for a subspace W of $\mathbf{R}^4$.

   Use the Gram-Schmidt process to convert S into an **orthogonal** basis for W, but **do not** bother to do the normalizing step.

1. Answer each question separately. Let A^T mean the transpose of A.

(a) We have $W_1, W_2 \leq V$ with $\dim(V) = 10$, $\dim(W_1) = 6$ and $\dim(W_2) = 7$. We know $W_1 + W_2 \leq V$ so $\dim(W_1 + W_2) \leq \dim(V) = 10$. Then the formula for the dimension of a sum of subspaces gives us $dim(W_1)+dim(W_2)-dim(W_1 \cap W_2) = 6+7-dim(W_1 \cap W_2) \leq 10$ so $3 \leq dim(W_1 \cap W_2)$. Since $W_1 \cap W_2 \leq W_1$, we get an upper bound of 6. Finally, all possibilities are $3 \leq dim(W_1 \cap W_2) \leq 6$.

(b) $\dim(Hom(F_3^2, F_4^5)) = (6)(20) = 120$. This is justified by the isomorphism $Hom(F_3^2, F_4^5) \cong F_6^{20}$ sending a linear transformation $L : F_3^2 \to F_4^5$ to the 20×6 matrix ${}_T[L]_S$ representing L from S to T.

(c) If $A^T = -A$ for $A \in \mathbf{R}_n^n$ where n is odd, then $det(A) = det(A^T) = det(-A) = (-1)^n det(A) = -det(A)$, so we can say that $det(A) = 0$.

(d) $det(A^T B^{-1} C^3) = \frac{det(A) det(C)^3}{det(B)} = \frac{(3)(2^3)}{-7} = \frac{24}{-7}$.

(e)
$$\det \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} = \det \begin{bmatrix} -2 & 0 & 0 & 8 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 2 \\ 5 & -5 & -5 & -7 \end{bmatrix} = 2^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 5 & -5 & -5 & -7 \end{bmatrix}$$
$$= 2^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -5 & -5 & 13 \end{bmatrix} = 2^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -5 & 8 \end{bmatrix} = 2^3 \det \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$
$$= 2^3\ (-1)^3\ 3 = -24.$$

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2. Let $\langle \cdot, \cdot \rangle : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be defined by $\langle X, Y \rangle = X^T M Y$ for a matrix $M \in \mathbf{R}_n^n$.

(a) Prove that $\langle X, Y \rangle$ is **bilinear**.

Solution: $\langle aX + bY, Z \rangle = (aX + bY)^T M Z = a(X^T M Z) + b(Y^T M Z) = a\langle X, Z \rangle + b\langle Y, Z \rangle$. Also, $\langle X, aY + bZ \rangle = X^T M(aY + bZ) = a(X^T M Y) + b(X^T M Z) = a\langle X, Y \rangle + b\langle X, Z \rangle$.

(b) Prove that $\langle X, Y \rangle = \langle Y, X \rangle$ iff $M = M^T$ is symmetric.

Solution: If $\langle X, Y \rangle = \langle Y, X \rangle$ for all $X, Y \in \mathbf{R}^n$ then $X^T M Y = Y^T M X$. But this is a 1×1 matrix, so the right side equals its transpose, giving $X^T M Y = (Y^T M X)^T = X^T M^T Y$. If we use standard basis vectors $X = \mathbf{e}_i$ and $Y = \mathbf{e}_j$, we get $\mathbf{e}_i^T M \mathbf{e}_j = \mathbf{e}_i^T M^T \mathbf{e}_j$ which means the (i, j)-entry of M equals the (i, j)-entry of M^T so $M = M^T$.

Conversely, if $M = M^T$ then $\langle X, Y \rangle = X^T M Y = (X^T M Y)^T = Y^T M^T X = Y^T M X = \langle Y, X \rangle$.

(c) Find the condition on $A \in \mathbf{R}_n^n$ which means $\langle X, Y \rangle = \langle AX, AY \rangle$ for all $X, Y \in \mathbf{R}^n$.

Solution: We require that $X^T M Y = (AX)^T M(AY)$, that is, $X^T M Y = X^T (A^T M A) Y$ for all $X, Y \in \mathbf{R}^n$. Using $X = \mathbf{e}_i$ and $Y = \mathbf{e}_j$ as in part (b), we find that the (i, j)-entries match in M and $A^T M A$ for all $1 \leq i, j \leq n$, so $M = A^T M A$ is the desired condition on A.

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3. The set $S = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for a subspace W of $\mathbf{R}^4$.

   Use the Gram-Schmidt process to convert S into an **orthogonal** basis for W, but **do not** bother to do the normalizing step.

   Solution: First let $w_1' = w_1$, then

$$w_2' = w_2 - \frac{w_2 \cdot w_1'}{w_1' \cdot w_1'} w_1' = w_2 - \frac{12}{4}\, w_1' = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}. \text{ Check that } w_1' \cdot w_2' = 0.$$

Let

$$w_3' = w_3 - \frac{w_3 \cdot w_1'}{w_1' \cdot w_1'} w_1' - \frac{w_3 \cdot w_2'}{w_2' \cdot w_2'} w_2' = w_3 - \frac{3}{4}\, w_1' - \frac{1}{4}\, w_2' = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

We may take any multiples of these vectors and they will still be orthogonal, so we can use $u_3 = 2w_3' = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Finally check that

$$S = \left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is an orthogonal set which also spans W, so is an orthogonal basis for W obtained by the Gram-Schmidt orthogonalization process.
