A very important use for a basis of a vector space V is to give "coordinates" for each $v \in V$.

<u>Def</u>. Let $S = \{v_1, \ldots, v_n\}$ be a basis for V, so $\forall v \in V$, $\exists a_1, \ldots, a_n \in F$ s.t. $v = \sum_{j=1}^{n} a_j v_j$. It is understood that S is actually an ordered list, so the corresponding list of scalar coefficients, $a_1, \ldots, a_n \in F$ is uniquely determined by v. So we define the "coordinate vector of v w.r.t. S" to be

$$[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n.$$

<u>Th</u>: The map $[\cdot]_S : V \to F^n$ is linear and bijective, so it is invertible and an isomorphism

Pf. Suppose $v, w \in V$ with

$v = \sum_{j=1}^{n} a_j v_j$ and $w = \sum_{j=1}^{n} b_j v_j$, so $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and

$[w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. Then $v + w = \sum_{j=1}^{n} (a_j + b_j) v_j$ so

$$[v+w]_S = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [v]_S + [w]_S .$$

For any $\alpha \in F$, $\alpha v = \sum_{j=1}^{n} (\alpha a_j) v_j$ so $[\alpha v]_S = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} =$

$\alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha [v]_S$. This shows $[\cdot]_S$ is <u>linear</u>.

$Ker([\cdot]_S) = \{v \in V / [v]_S = 0_1^n\}$ so $v \in Ker([\cdot]_S)$ iff

$v = \sum_{j=1}^{n} 0 v_j = \theta_V$ so the map is <u>injective</u>.

$\forall \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n$, the vector $v = \sum_{j=1}^{n} a_j \cdot v_j \in V$

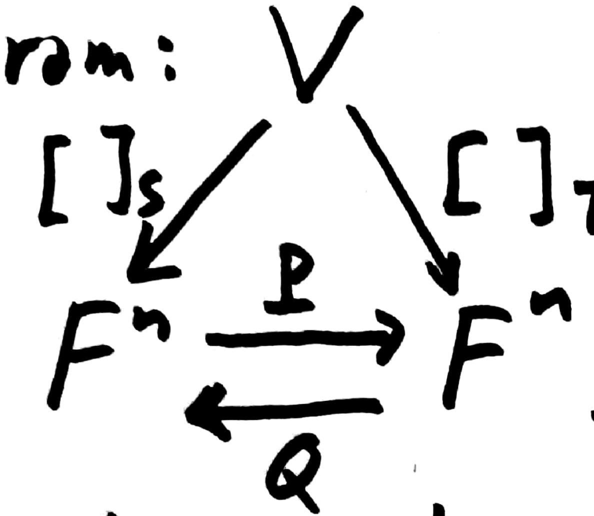
has $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, so $[\cdot]_S$ is <u>surjective</u>. This <u>gives the rest of the properties, invert, isom.</u> □

<u>Note</u>: For $1 \le j \le n$, $\forall v_j \in S$, $[v_j]_S = e_j$ is the j^{th} standard basis vector of F^n, so

$\{[v_1]_S, [v_2]_S, \ldots, [v_n]_S\}$ is the <u>std.</u> basis of F^n.

Fundamental Question about coordinates:
If V has two bases, $S = \{v_1, \ldots, v_n\}$ and $T = \{w_1, \ldots, w_n\}$, how are $[v]_S$ and $[v]_T$ related?

Diagram:



Can we find $P \in F^n_n$ s.t. $\forall v \in V$, $P[v]_S = [v]_T$ and $Q \in F^n_n$ s.t. $Q[v]_T = [v]_S$

The maps on the bottom lines are actually L_P and L_Q, but we just label the arrows with P and Q.

What would these matrices have to be if they existed? <u>Answer</u>: Certainly we would need $P[v_j]_S = [v_j]_T$ and $Q[w_j]_T = [w_j]_S$ for $1 \le j \le n$. But $[v_j]_S = e_j = [w_j]_T$ so $Col_j(P) = [v_j]_T$ and $Col_j(Q) = [w_j]_S$.

Thus, to find P, find its columns, which are the coordinates w.r.t T of the basis S vectors. Similarly, the columns of Q are the coordinates w.r.t. S of the basis T vectors.

How do we find the coordinates of a vector $v \in V$ w.r.t. a basis, S or T, of V?

Solve a lin. sys., of cource!

$$\sum_{j=1}^{n} x_j \cdot v_j = v$$ is solved by row reducing

$$\underset{\text{as columns}}{[S \mid v]} \xrightarrow{\text{r.r.}} [I_n \mid [v]_S]$$

$$\sum_{j=1}^{n} x_j \cdot w_j = v$$ is solved by $$\underset{\text{as columns}}{[T \mid v]} \xrightarrow{\text{r.r.}} [I_n \mid [v]_T]$$

So to get P s.t. $P[v_j]_S = [v_j]_T$ we need to solve n linear systems, $1 \le j \le n$, but all have the same coeff. matrix, so solve all at once:

① $[T|S] \xrightarrow{r.r.} [I_n | {}_TP_S]$ gives the "transition matrix" ${}_TP_S$ from S to T s.t. ${}_TP_S[v]_S = [v]_T$

(as columns)

② $[S|T] \xrightarrow{r.r.} [I_n | {}_SQ_T]$ gives the "transition matrix" ${}_SQ_T$ from T to S s.t. ${}_SQ_T[v]_T = [v]_S$

(as columns)

__Th__: If $S = \{v_1, \dots, v_n\}$ and $T = \{w_1, \dots, w_n\}$ are bases of V, then $\exists\ {}_TP_S, {}_SQ_T \in F_n^n$ s.t. $\forall v \in V$, ${}_TP_S[v]_S = [v]_T$ and ${}_SQ_T[v]_T = [v]_S$.

Pf. We have an algorithm by row reduction guaranteed to find the matrices P and Q s.t. the formulas are true for $v = v_j \in S$ in the equation $P[v_j]_S = [v_j]_T$ and for $v = w_j \in T$ for $Q[w_j]_T = [w_j]_S$. But $\forall v \in V$, can write

$$v = \sum_{j=1}^{n} a_j v_j \text{ so } P[v]_S = P\left[\sum_{j=1}^{n} a_j v_j\right]_S = \sum_{j=1}^{n} a_j P[v_j]_S$$

(by lin. of $[\cdot]_S$) $= \sum_{j=1}^{n} a_j [v_j]_T = \left[\sum_{j=1}^{n} a_j v_j\right]_T$ (by lin. of $[\cdot]_T$)

$= [v]_T$ so $P[v]_S = [v]_T$, $\forall v \in V$.

The argument for any $w = \sum_{j=1}^{n} b_j w_j \in V$ is similar. □

<u>Cor</u>: If ${}_T P_S [v]_S = [v]_T$ and ${}_S Q_T [v]_T = [v]_S$ then ${}_T P_S = {}_S Q_T^{-1}$.

<u>Pf</u>. By substitution: ${}_T P_S ({}_S Q_T [v]_T) = [v]_T$ and ${}_S Q_T ({}_T P_S [v]_S) = [v]_S$. By assoc. of matrix mult. and using $v = w_j \in T$ in the first eq. and $v = v_j \in S$ in the second eq., we get

$$({}_T P_S \, {}_S Q_T) e_j = e_j \text{ and } ({}_S Q_T \, {}_T P_S) e_j = e_j, \quad 1 \le j \le n.$$

So ${}_T P_S \, {}_S Q_T = I_n = {}_S Q_T \, {}_T P_S$. □

Usually I will write ${}_T P_S [v]_S = [v]_T$ and

$${}_S P_T [v]_T = [v]_S.$$

Another way to see why these two transition matrices are inverses of each other is to look at the algorithms for finding them.

Since $[T \mid S] \xrightarrow{①} [I_n \mid {}_TP_S]$ and $[S \mid T] \xrightarrow{②} [I_n \mid {}_SP_T]$

we could start with ↑ switching the two sides:

$$[{}_TP_S \mid I_n] \xrightarrow[\text{steps of ①}]{\text{reverse}} [S \mid T] \xrightarrow{②} [I_n \mid {}_SP_T]$$ says

$${}_SP_T = ({}_TP_S)^{-1}.$$

<u>Example</u>: Let $V = \mathbb{R}^2_2$, $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ the <u>std</u>. basis, $T = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ (see page 99)

For $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^2_2$ find $[v]_S$ and $[v]_T$ and the transition mat's: ${}_TP_S$ and ${}_SP_T$.

To find $[v]_S \in \mathbb{R}^4$, solve $\sum_{j=1}^{4} x_j v_j = v$ with $S = \{v_1, v_2, v_3, v_4\}$, get lin. sys. $[S \mid v]$ = as col's

$$\left[\begin{array}{cccc|c} 1&0&0&0&a\\ 0&1&0&0&b\\ 0&0&1&0&c\\ 0&0&0&1&d \end{array}\right] \quad \begin{array}{l} x_1 = a\\ x_2 = b\\ x_3 = c\\ x_4 = d \end{array} \quad \text{so } [v]_S = \begin{bmatrix} a\\ b\\ c\\ d \end{bmatrix} \in \mathbb{R}^4.$$

To find $[v]_T \in \mathbb{R}^4$, solve $\sum_{j=1}^{4} x_j w_j = v$ with $T = \{w_1, w_2, w_3, w_4\}$, get lin. sys. $[T \mid v]$ = as col's

$$\left[\begin{array}{cccc|c} 1&1&1&1&a\\ 1&1&1&0&b\\ 1&1&0&0&c\\ 1&0&0&0&d \end{array}\right] \xrightarrow{\text{r.r.}} \left[\begin{array}{cccc|c} 1&0&0&0&d\\ 0&1&0&0&c-d\\ 0&0&1&0&b-c\\ 0&0&0&1&a-b \end{array}\right] \quad \begin{array}{l} x_1 = d\\ x_2 = c-d\\ x_3 = b-c\\ x_4 = a-b \end{array} \quad \text{so } [v]_T = \begin{bmatrix} d\\ c-d\\ b-c\\ a-b \end{bmatrix}$$

and we can check: $d\begin{bmatrix} 1&1\\ 1&1 \end{bmatrix} + (c-d)\begin{bmatrix} 1&1\\ 1&0 \end{bmatrix} + (b-c)\begin{bmatrix} 1&1\\ 0&0 \end{bmatrix} + (a-b)\begin{bmatrix} 1&0\\ 0&0 \end{bmatrix} = \begin{bmatrix} a&b\\ c&d \end{bmatrix}$

But notice: $[v]_T = \begin{bmatrix} d \\ c-d \\ b-c \\ a-b \end{bmatrix} = \underbrace{\begin{bmatrix} 0&0&0&1 \\ 0&0&1&-1 \\ 0&1&-1&0 \\ 1&-1&0&0 \end{bmatrix}}_{{}_T P_S} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[v]_S}$

and by the algorithm:

$$\underset{\text{as col's}}{[T|S]} = \left[\begin{array}{cccc|cccc} 1&1&1&1&1&0&0&0 \\ 1&1&1&0&0&1&0&0 \\ 1&1&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&1 \end{array}\right] \xrightarrow{\text{r.r.}} \left[\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1&-1 \\ 0&0&1&0&0&1&-1&0 \\ 0&0&0&1&1&-1&0&0 \end{array}\right]$$

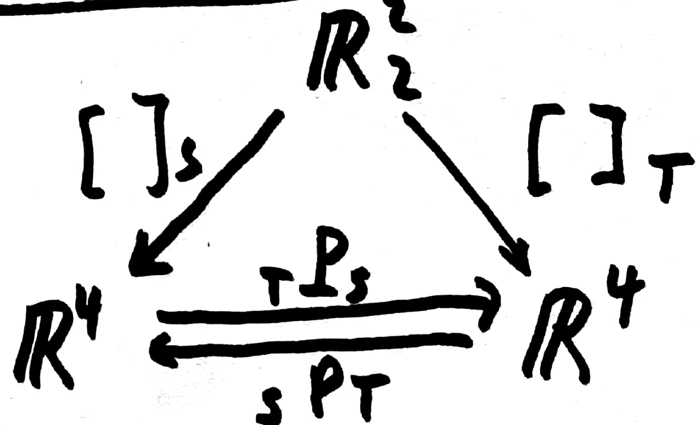
(the right-hand block is ${}_T P_S$)

while

$$\underset{\text{as col's}}{[S|T]} = \left[\begin{array}{cccc|cccc} 1&0&0&0&1&1&1&1 \\ 0&1&0&0&1&1&1&0 \\ 0&0&1&0&1&1&0&0 \\ 0&0&0&1&1&0&0&0 \end{array}\right]$$

is already in RREF on leftside so (the right-hand block) $= {}_S P_T$.

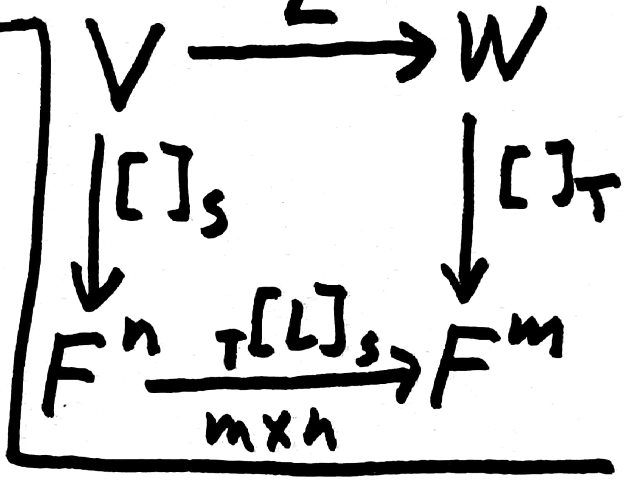
Diagram:



Th: Let $L: V \to W$ be linear, $S = \{v_1, \dots, v_n\}$ a basis of V, $T = \{w_1, \dots, w_m\}$ a basis of W. Then we can find $A = {}_T[L]_S \in F^m_n$ such that

$$\forall v \in V, \quad {}_T[L]_S \, [v]_S = [L(v)]_T .$$

Pf. The appropriate diagram is:



What would ${}_T[L]_S$ have to be if the equation is true for all $v_j \in S$?

$${}_T[L]_S \, [v_j]_S = {}_T[L]_S \, e_j = Col_j({}_T[L]_S) = [L(v_j)]_T$$

so the columns of ${}_T[L]_S$ are the coordinates w.r.t. T of the n images $L(v_j) \in W$. This gives the following algorithm to find it:

$[T \mid L(S)] \xrightarrow{r.r.} [I_m \mid {}_T[L]_S]$ and this can always be done since T is a basis of W so $T \underset{row}{\sim} I_m$ (T vectors written as col's). 122

as columns

Then $\forall v \in V$, write $v = \sum_{j=1}^{n} a_j v_j$ and check

$${}_T[L]_S [v]_S = {}_T[L]_S \left[\sum_{j=1}^{n} a_j v_j\right]_S = \sum_{j=1}^{n} a_j \, {}_T[L]_S [v_j]_S =$$

$$\sum_{j=1}^{n} a_j [L(v_j)]_T = \left[\sum_{j=1}^{n} a_j L(v_j)\right]_T = \left[L\left(\sum_{j=1}^{n} a_j v_j\right)\right]_T$$

$$= [L(v)]_T. \quad \square$$

<u>Def</u>. The matrix ${}_T[L]_S \in F_n^m$ is called the matrix representing L from S to T (w.r.t. S and T).

(Any $L : V \to W$ has been related to an $L_A : F^n \to F^m$.)