**Example:** Let $L : \mathbb{R}^2_2 \longrightarrow \mathbb{R}^3$ be the linear map

$$L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a+2b+c-d \\ a+b+2c+d \\ a-b+c-d \end{bmatrix}.$$

Let $S = \{v_1, v_2, v_3, v_4\}$ be the std. basis of $\mathbb{R}^2_2$, $T = \{w_1, w_2, w_3\}$ be the std. basis of $\mathbb{R}^3$,

$$S' = \left\{ v_1' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, v_2' = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, v_3' = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, v_4' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

another basis of $\mathbb{R}^2_2$ and

$$T' = \left\{ w_1' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, w_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, w_3' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

another basis of $\mathbb{R}^3$.

① Find ${}_T[L]_S$ ② Find ${}_{T'}[L]_{S'}$

③ Find ${}_{S'}P_S, {}_SP_{S'} \in \mathbb{R}^4_4$ transition matrices.

④ Find ${}_{T'}Q_T, {}_TQ_{T'} \in \mathbb{R}^3_3$ transition matrices.

⑤ What are the relations among these matrices?

## One Diagram to Rule them All:

$$\begin{tikzcd}
 & \mathbb{R}^2_2 \arrow[r, "L"] \arrow[d, "{[\;]_S}"] \arrow[ddl, "{[\;]_{S'}}"'] & \mathbb{R}^3 \arrow[d, "{[\;]_T}"'] \arrow[ddr, "{[\;]_{T'}}"] & \\
 & \mathbb{R}^4 \arrow[r, "{{}_T[L]_S}", "3\times 4"'] \arrow[dl, shift left, "{{}_{S'}P_S}"] & \mathbb{R}^3 \arrow[dr, shift left, "{{}_{T'}Q_T}", "3\times 3"'] & \\
\mathbb{R}^4 \arrow[rrr, "{{}_{T'}[L]_{S'}}", "3\times 4"'] \arrow[ur, shift left, "{{}_S P_{S'}}", "4\times 4"'] & & & \mathbb{R}^3 \arrow[ul, shift left, "{{}_T Q_{T'}}"]
\end{tikzcd}$$

$${}_T[L]_S\,[v]_S = [L(v)]_T \qquad {}_S P_{S'}\,[v]_{S'} = [v]_S$$

$${}_{T'}[L]_{S'}\,[v]_{S'} = [L(v)]_{T'} \qquad {}_{T'}Q_T\,[w]_T = [w]_{T'}$$

① To find ${}_T[L]_S$ first compute $L(S)$: 

$L(v_1) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $L(v_2) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $L(v_3) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $L(v_4) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

Solve

$[T \mid L(S)] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 & -1 \end{array}\right]$ already RREF so ${}_T[L]_S = \begin{bmatrix} -1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

__Check:__ ${}_T[L]_S [v]_S = \begin{bmatrix} -1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -a+2b+c-d \\ a+b+2c+d \\ a-b+c-d \end{bmatrix}$

$= [L(v)]_T$ since for

$w = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, $w = x\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{w_1} + y\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{w_2} + z\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{w_3}$ so $[w]_T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = w$

__Note:__ ${}_T[L]_S$ could be "read off" the formula for $L(v)$ since $S$ and $T$ were std. bases.

② To find ${}_{T'}[L]_{S'}$ "directly" from the algorithm (without using transition matrices), first find $L(S')$:

$$L(v_1') = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \quad L(v_2') = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \quad L(v_3') = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad L(v_4') = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Solve

$$[T' \mid L(S')] = \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 2 & 1 & -1 \\ 2 & 1 & 1 & 5 & 4 & 2 & 1 \\ 3 & 2 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 & 0 & 3 \\ 0 & 2 & -3 & -3 & -5 & -3 & 4 \end{array}\right]$$

$-2 \quad 0 \quad -2 \quad -2 \quad -4 \quad -2 \quad 2$

$-3 \quad 0 \quad -3 \quad -3 \quad -6 \quad -3 \quad 3$

$0 \quad -2 \quad 2 \quad -6 \quad 0 \quad 0 \quad -6$

$$\to \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 & 0 & 3 \\ 0 & 0 & -1 & -9 & -5 & -3 & -2 \end{array}\right] \to \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -8 & -3 & -2 & -3 \\ 0 & 1 & 0 & 12 & 5 & 3 & 5 \\ 0 & 0 & 1 & 9 & 5 & 3 & 2 \end{array}\right]$$

gives ${}_{T'}[L]_{S'} \in \mathbb{R}^3_4$.

To check ${}_{T'}[L]_{S'}[v]_{S'} = [L(v)]_{T'}$ we need to [12] find $[v]_{S'}$: Solve $\sum_{j=1}^{4} x_j \cdot v_j' = v$ by row reducing

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 1 & 1 & 1 & 0 & b \\ 1 & 1 & 0 & 0 & c \\ 1 & 0 & 0 & 0 & d \end{array}\right]}_{S' \quad v} \to \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 0 & 1 & 0 & 0 & c-d \\ 0 & 0 & 1 & 0 & b-c \\ 0 & 0 & 0 & 1 & a-b \end{array}\right]}_{I_4 \quad [v]_{S'}}$$

was done on page 119 (but we called that basis $T$ on p. 119).

$$\underbrace{\begin{bmatrix} -8 & -3 & -2 & -3 \\ 12 & 5 & 3 & 5 \\ 9 & 5 & 3 & 2 \end{bmatrix}}_{{}_{T'}[L]_{S'}} \underbrace{\begin{bmatrix} d \\ c-d \\ b-c \\ a-b \end{bmatrix}}_{[v]_{S'}} = \begin{bmatrix} -3a+b-c-5d \\ 5a-2b+2c+7d \\ 2a+b+2c+4d \end{bmatrix} \stackrel{?}{=} [L(v)]_{T'}$$

The easiest check is: $L(v) \stackrel{?}{=}$

$$(-3a+b-c-5d)\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{w_1'} + (5a-2b+2c+7d)\underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{w_2'} + (2a+b+2c+4d)\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{w_3'} =$$

$$\begin{bmatrix} -a+2b+c-d \\ a+b+2c+d \\ a-b+c-d \end{bmatrix} = L(v) \text{ so that was } [L(v)]_{T'}.$$

③ We already found on p. 119-120 that

$$ {}_S P_{S'} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } {}_{S'}P_S = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = {}_S P_{S'}^{-1}.$$

④
$$\begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 & 2 & 0 \end{array}\right] \\ \quad T \qquad\qquad T' \end{array} \text{ is in RREF so } {}_T Q_{T'} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\begin{array}{c} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \\ \quad T' \qquad\qquad T \end{array} \xrightarrow{\text{r.r.}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \rightarrow$$

$-2 \; 0 \; -2 \; -2 \; 0 \; 0$

$-3 \; 0 \; -3 \; -3 \; 0 \; 0$

$0 \; -2 \; 2 \; 4 \; -2 \; 0$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] {}_{T'}Q_T$$

⑤ The relations among transition mat's are known: ${}_{S}P_{S'}^{-1} = {}_{S'}P_{S}$ and ${}_{T}Q_{T'}^{-1} = {}_{T'}Q_{T}$.

The bottom rectangle of the diagram says:

$$\underset{3\times 4}{{}_{T'}[L]_{S'}} = \underset{3\times 3}{{}_{T'}Q_{T}}\ \underset{3\times 4}{{}_{T}[L]_{S}}\ \underset{4\times 4}{{}_{S}P_{S'}}$$

so let's check:

$$\begin{bmatrix} 2 & -2 & 1 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

checks!

$$= \begin{bmatrix} -3 & 1 & -1 & -5 \\ 5 & -2 & 2 & 7 \\ 2 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -8 & -3 & -2 & -3 \\ 12 & 5 & 3 & 5 \\ 9 & 5 & 3 & 2 \end{bmatrix} = {}_{T'}[L]_{S'}$$

This example illustrates a general Theorem.

<u>Th.</u> Let $L: V \to W$, $S = \{v_1, \dots, v_n\}$ and $S' = \{v_1', \dots, v_n'\}$ bases of $V$, $T = \{w_1, \dots, w_m\}$ and $T' = \{w_1', \dots, w_m'\}$ bases of $W$, ${}_T[L]_S \in F_n^m$ the matrix rep'ing $L$ from $S$ to $T$, ${}_{T'}[L]_{S'} \in F_n^m$ the matrix rep'ing $L$ from $S'$ to $T'$, and let

${}_S P_{S'} \in F_n^n$ be the transition matrix from $S'$ to $S$,

${}_{T'} Q_T \in F_m^m$ " " " " $T$ to $T'$.

Then $${}_{T'}[L]_{S'} = \underset{m \times m}{{}_{T'}Q_T}\ \underset{m \times n}{{}_T[L]_S}\ \underset{n \times n}{{}_S P_{S'}}.$$

<u>Pf.</u> We have the diagram and equations defining these matrices as follows:

$$
\begin{tikzcd}
& V \arrow[r, "L"] \arrow[d, "{[\,]_S}"] \arrow[ddl, "{[\,]_{S'}}"'] & W \arrow[d, "{[\,]_T}"] \arrow[ddr, "{[\,]_{T'}}"] & \\
& F^n \arrow[r, "{{}_T[L]_S}"] & F^m \arrow[dr, "{{}_{T'}Q_T}"] & \\
F^n \arrow[ur, "{{}_SP_{S'}}"] \arrow[rrr, "{{}_{T'}[L]_{S'}}"'] & & & F^m
\end{tikzcd}
$$

**Equations**

1. ${}_T[L]_S[v]_S = [L(v)]_T$
2. ${}_{T'}[L]_{S'}[v]_{S'} = [L(v)]_{T'}$
3. ${}_SP_{S'}[v]_{S'} = [v]_S$
4. ${}_{T'}Q_T[w]_T = [w]_{T'}$

So $\forall v \in V$,

$${}_{T'}[L]_{S'}[v]_{S'} \overset{②}{=} [L(v)]_{T'} \overset{④}{=} {}_{T'}Q_T[L(v)]_T \overset{①}{=} {}_{T'}Q_T\,{}_T[L]_S[v]_S$$

$$\overset{③}{=} {}_{T'}Q_T\,{}_T[L]_S\,{}_SP_{S'}[v]_{S'}.$$

Using $v = v_j' \in S'$, have $[v_j']_{S'} = e_j$ std. basis vector in $F^n$, get for $1 \le j \le n$,

$$Col_j({}_{T'}[L]_{S'}) = Col_j({}_{T'}Q_T\,{}_T[L]_S\,{}_SP_{S'}),$$

so the whole matrices are equal. $\square$

<u>Question</u>: If you had any choice of bases $S'$ in $V$ and $T'$ in $W$, what would be the "nicest" matrix ${}_{T'}[L]_{S'}$ rep'ing $L$?

<u>Alternative</u>: If you had any choice of invertible matrices $Q \in F^m_m$ and $P \in F^n_n$ what is the "nicest" matrix $B = QAP$ you could get from a given $A \in F^m_n$?

<u>Answer</u>: We can interpret $Q$ as doing row op's to $A$, and $P$ as doing col. op's to $QA$. The "best" $QA$ is in RREF. Allowing col. op's on $QA$ can give Block Identity Form (BIF)

$$B = QAP = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \text{ where } r = \operatorname{rank}(A).$$

From the point of view of linear maps and bases and matrices rep'ing maps, we can see it as follows:

For $L: V \to W$ let $K = Ker(L)$ have basis $\{k_1, \dots, k_{n-r}\}$ for $r = rank(A) = dim(Range(L))$

Extend this to a basis of $V$ but add the new vectors at the beginning of the list; get $S' = \{v_1', \dots, v_r', v_{r+1}' = k_1, \dots, v_n' = k_{n-r}\}$. Then

$L(S') = \{w_1' = L(v_1'), \dots, w_r' = L(v_r'), \underbrace{\theta_w = L(k_1), \dots, \theta_w = L(k_{n-r})}_{\text{redundant}}\}$

get a basis for Range(L) to be $\{w_1', \dots, w_r'\}$ and can extend it to a basis $T'$ for $W$, $T' = \{w_1', \dots, w_r', \dots w_m'\}$ any way you like.

To find ${}_{T'}[L]_{S'}=B$, find $[L(v_j')]_{T'}=\mathrm{Col}_j(B)$.
for $1\le j\le r$, $L(v_j')=w_j'$ so $[L(v_j')]_{T'}=e_j\in F^m$
but for $r<j\le n$, $L(v_j')=\theta_W$ since $v_j'=k_{j-r}\in K$
so $[L(v_j')]_{T'}=0_1^m\in F^m$.
This gives ${}_{T'}[L]_{S'}=B=\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]\in F_n^m$ in BIF.

---

<u>Th</u>: Let $L:V\to W$ for $\dim(V)=n<\infty$. Then
$\dim(V)=\dim(\mathrm{Ker}(L))+\dim(\mathrm{Range}(L))$.
<u>Pf</u>. Let $B=\{v_1,\dots,v_k\}$ be a basis for $\mathrm{Ker}(L)$, $k\ge 0$.
Extend $B$ to a basis $S=\{v_1,\dots,v_k,\dots,v_n\}$ for all of $V$.
Then $\mathrm{Range}(L)=\left\{\sum_{j=1}^{n} a_jL(v_j)\in W \,\middle|\, a_j\in F\right\}=\langle L(S)\rangle$, but

$L(B) = \{L(v_1), \ldots, L(v_k)\} = \{\theta_W, \ldots, \theta_W\}$ so the vectors $L(v_j) = \theta_W$ for $1 \leq j \leq k$ are redundant in $L(S)$ and $\text{Range}(L) = \langle L(v_{k+1}), \ldots, L(v_n)\rangle$ is spanned by just $n-k$ vectors. <u>Claim</u>: These $n-k$ vectors are indep.

<u>Pf</u>. If $\sum_{j=k+1}^{n} a_j L(v_j) = \theta_W$ then $L\left(\sum_{j=k+1}^{n} a_j v_j\right) = \theta_W$

so $\sum_{j=k+1}^{n} a_j v_j \in Ker(L) = \langle B \rangle$ so

$= \sum_{j=1}^{k} b_j v_j$ giving $\theta_V = \sum_{j=1}^{k} -b_j v_j + \sum_{j=k+1}^{n} a_j v_j$ a

dep. relation on basis $S$, so all its coeff's are $0$.

$\dim(\text{Range}(L)) = n-k$ and $\dim(Ker(L)) = k$ so

they add up to $n = \dim(V)$. $\square$

## Applications of the dim formula (rank/nullity)



<u>Def.</u> $Nul(A) = Ker(L_A)$ for $A \in F^m_n$

$Nullity(A) = \dim(Nul(A)) = n-r$ if $rank(A) = r$

$= \dim(Range(L_A))$.

<u>Th</u>: Suppose $L : V \to W$, $\dim(V) = n$, $\dim(W) = m$.

(a) If $n > m$ then $\dim(Ker(L)) \geq n-m > 0$.

(b) If $n < m$ then $\dim(Range(L)) \leq n < m$.

(c) If $n = m$ then $L$ inj $\Leftrightarrow$ $L$ surj

---

<u>Pf.</u> (a) $n = \dim(Ker(L)) + \dim(Range(L))$ but

$0 \leq \dim(Range(L)) \leq m$ so

$\dim(Ker(L)) = n - \dim(Range(L)) \geq n-m > 0$ if $n > m$.

(b) $\dim(Ker(L)) \geq 0$ so $n \geq \dim(Range(L))$ is always true.

(c) Suppose $n=m$. ($\Rightarrow$) If $L$ is inj then $\dim(\mathrm{Ker}(L))=0$ so $m=n=\dim(\mathrm{Range}(L))$ gives $\dim(W)=\dim(\mathrm{Range}(L))$. For $W$ fin. dim'l, get $W=\mathrm{Range}(L)$ so $L$ is surj.

($\Leftarrow$) Suppose $L$ is surj. So $\mathrm{Range}(L)=W$ so $\dim(\mathrm{Range}(L))=\dim(W)=m=n$ so $n=\dim(\mathrm{Ker}(L))+n$ gives $\dim(\mathrm{Ker}(L))=0$ so $L$ is inj. $\square$

---

Ex: For $L: F_5^2 \to F^7$ what are all possible values of $\dim(\mathrm{Ker}(L))$?

$$10=\dim(F_5^2)=\dim(\mathrm{Ker}(L))+\dim(\mathrm{Range}(L))$$

Since $0 \le \dim(\mathrm{Range}(L)) \le 7$ we must have $3 \le \dim(\mathrm{Ker}(L)) \le 10$.