Let's study the special case of $L \in \mathrm{End}(V)$

$$\begin{tikzcd}
& V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] \arrow[ddl, "{[\ ]_T}"'] & V \arrow[d, "{[\ ]_S}"'] \arrow[ddr, "{[\ ]_T}"] & \\
& F^n \arrow[r, "{{}_S[L]_S}", "n\times n"'] & F^n \arrow[dr, "{{}_TP_S}", "n\times n"'] & \\
F^n \arrow[ur, "{{}_SP_T}", "n\times n"'] \arrow[rrr, "{{}_T[L]_T}", "n\times n"'] & & & F^n
\end{tikzcd}$$

Let $S=\{v_1,\dots,v_n\}$ and $T=\{w_1,\dots,w_n\}$ be bases of V, ${}_TP_S, {}_SP_T \in F^n_n$ the transition matrices s.t. ${}_TP_S = {}_SP_T^{-1}$. Then we have from the general Theorem,

$${}_T[L]_T = {}_TP_S\,{}_S[L]_S\,{}_SP_T$$

but the relation between $B={}_T[L]_T$ and $A={}_S[L]_S$ is that

$$B = P^{-1}AP.$$

<u>Def</u>: Say $A, B \in F^n_n$ are <u>similar</u> when $B=P^{-1}AP$ for some invertible $P \in F^n_n$. write: $A \underset{\text{sim}}{\sim} B$

<u>Th.</u> $\underset{\text{sim}}{\sim}$ is an equivalence relation on F^n_n (reflexive, symmetric, transitive).

<u>Questions:</u> ① If you could choose any basis T of V, what choice would give "nice" ${}_T[L]_T$?

② If you could pick any invertible $P \in F^n_n$, what choice would give "nice" $B = P^{-1}AP$?

<u>Answers:</u> First try to find T or $P = {}_S P_T$ making ${}_T[L]_T$ and B <u>diagonal</u>, that is,

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

for some scalars $\lambda_1, \dots, \lambda_n \in F$ (not necessarily distinct).

That is the "diagonalization" topic in elem. lin. alg. courses. We will review it.

If L or A cannot be diagonalized, what are the next best options?

Two options: ① Jordan Canonical Form, ② Rational Canonical Form.

We will study both of these.

Th. Let $L: V \to V$, $S = \{v_1, \dots, v_n\}$ a basis of V, $A = {}_S[L]_S \in F_n^n$. For basis $T = \{w_1, \dots, w_n\}$ of V, ${}_T[L]_T = D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ iff $L(w_j) = \lambda_j \cdot w_j$ for $1 \le j \le n$.

Pf. $\mathrm{Col}_j({}_T[L]_T) = [L(w_j)]_T = \mathrm{Col}_j(D) = \lambda_j \cdot e_j$ iff $L(w_j) = 0w_1 + \dots + \lambda_j \cdot w_j + 0w_{j+1} + \dots + 0w_n$ by def. of coordinates.

<u>Def.</u> For $L: V \to V$, say $\theta \neq v \in V$ is an <u>eigenvector</u> for L with <u>eigenvalue</u> $\lambda \in F$ when $L(v) = \lambda v$. For $A \in F^n_n$, say $0^n_1 \neq X \in F^n$ is an e.vect. for A with e.val. $\lambda \in F$ when $AX = \lambda X$.

<u>Def.</u> Say $L: V \to V$ is <u>diagonalizable</u> when there is some basis T of V for which ${}_T[L]_T = D$ is diagonal. Say $A \in F^n_n$ is diag.able when $\exists\, P \in F^n_n$ invertible s.t. $P^{-1}AP = D$ is diagonal.

<u>Th.</u> $L: V \to V$ is diag.able iff $\exists\, w_1, \dots, w_n \in V$ s.t. $T = \{w_1, \dots, w_n\}$ is a basis of V with each $w_j \in T$ an e.vector for L with some e.value λ_j.

Goals: ① Find an algorithm to find such an e-basis T, if possible, or prove one does not exist for the given $L: V \to V$. 142

② Find an algorithm to get an invertible $P \in F_n^n$ s.t. $P^{-1}AP = D$ is diag. if possible for the given $A \in F_n^n$.

Def. For $L: V \to V$, $\lambda \in F$, let $L_\lambda = \{v \in V \mid L(v) = \lambda v\}$. If $L_\lambda \neq \{\theta\}$, call it the λ e-space for L.
For $A \in F_n^n$, $\lambda \in F$, let $A_\lambda = \{X \in F^n \mid AX = \lambda X\}$. If $A_\lambda \neq \{0_1^n\}$ call it the λ e-space for A.

Note: $\forall \lambda \in F$, $L_\lambda \leq V$ and $A_\lambda \leq F^n$ since $\theta_V \in L_\lambda$, $0_1^n \in A_\lambda$, and both are closed under $+$ and $\cdot$.

For most $\lambda \in F$, $L_\lambda = \{\theta_V\}$ and $A_\lambda = \{0^n_1\}$ means there are no e.vectors for L and A with e.val. λ. How can we find the few special $\lambda \in F$ whose non-trivial e-spaces L_λ and A_λ contain e.vectors that can contribute to T or to P (as columns)?

Viewpoints on L_λ and A_λ which help:

$v \in L_\lambda$ iff $L(v) = \lambda v = \lambda I_V(v)$ iff $(L - \lambda I_V)(v) = \theta_V$ where we understand $L - \lambda I_V \in End(V) = Lin(V, V)$. So

$L_\lambda = Ker(L - \lambda I_V) = Ker(\lambda I_V - L)$

is an easy way to see that $L_\lambda \leq V$.

$X \in A_\lambda$ iff $AX = \lambda X = \lambda I_n X$ iff $(A-\lambda I_n)(X) = 0_1^n$ so $A_\lambda = Nul(A-\lambda I_n) = Nul(\lambda I_n - A) \leq F^n$.

We know how to use row reduction to find $Ker(L-\lambda I_V)$ and $Nul(A-\lambda I_n)$ but the issue is to find those $\lambda \in F$ s.t. these subspaces are <u>nontrivial</u>, that is, s.t. there is at least one free variable in the solution set. We do that for $Ker(L-\lambda I_V)$ by using the matrix $A = {}_S[L]_S$ so that

$${}_S[L-\lambda I_V]_S = {}_S[L]_S - \lambda\, {}_S[I_V]_S = A - \lambda I_n.$$

Th: Let $L: V \to V$ and $A = {}_S[L]_S$ for some basis S of V. Then $\lambda \in F$ is an e.value for L and for A iff $\operatorname{rank}(A - \lambda I_n) < n$.

For small n it is not hard to check this condition directly, as shown in the examples below. But the usual method uses det, and gives us the important concept of characteristic poly.

Example: Let $L: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be

$$L\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then for $S = \{e_1, e_2, \dots, e_n\}$ std basis of $\mathbb{R}^n$,

${}_S[L]_S = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = D$ is diagonal so L is diag-able. $L(e_j) = \lambda_j e_j$ for $1 \le j \le n$, works for <u>any</u> choices of $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Special case: $\lambda_1 = \cdots = \lambda_n = c$, $D = cI_n$ is a <u>scalar</u> matrix, $L = c\, I_{\mathbb{R}^n}$ is a scalar operator on $V = \mathbb{R}^n$.

Example: $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$
$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ so ${}_S[L]_S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (for S std basis)
is not diagonal. $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} = w_1, \begin{bmatrix} 1 \\ -1 \end{bmatrix} = w_2 \right\}$

$L\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ means w_1 is an e-vector
for this L with e-value $\lambda_1 = 1$.

$L\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ means w_2 is an e-vector
for this L with e-value $\lambda_2 = -1$.

${}_T[L]_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D$ is diagonal
so this L is diag-able.

Problem: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^2_2$, and let $L = L_A : \mathbb{R}^2 \to \mathbb{R}^2$ so $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}$

Is L diag-able? What would it mean for $w = \begin{bmatrix} x \\ y \end{bmatrix}$ to be an e-vector for L with e-value $\lambda \in \mathbb{R}$? It would mean that

$$\begin{bmatrix} x+y \\ x+y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} \text{ iff } \begin{matrix} x+y = \lambda x \\ x+y = \lambda y \end{matrix} \ \underline{\text{and}}$$

so $(1-\lambda)x + y = 0$ and $x + (1-\lambda)y = 0$ so $\left[\begin{array}{cc|c} (1-\lambda) & 1 & 0 \\ 1 & (1-\lambda) & 0 \end{array}\right]$

would have to have a <u>non-trivial</u> solution.

Have two approaches:

Row reduction:

$$\begin{bmatrix} (1-\lambda) & 1 & 0 \\ 1 & (1-\lambda) & 0 \end{bmatrix} \to \begin{bmatrix} -\lambda & \lambda & 0 \\ 1 & 1-\lambda & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } \lambda = 0$$

$-1 \qquad \lambda - 1$

but if $\lambda \neq 0$, get $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1-\lambda & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2-\lambda & 0 \end{bmatrix}$

for $\lambda \neq 2$ this has only the trivial solution.

For $\lambda = 2$ get $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = x_2 = r$, $x_2 = r \in \mathbb{R}$ free

For $\lambda = 0$ get $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -x_2 = -r$, $x_2 = r \in \mathbb{R}$ free

Let $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $L\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $L\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} = w_1, \begin{bmatrix} 1 \\ -1 \end{bmatrix} = w_2 \right\}$ basis of $\mathbb{R}^2$

$L(w_1) = 2w_1$ and $L(w_2) = \theta = 0 \cdot w_2$

${}_T[L]_T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is <u>diagonal</u>.

<u>Second approach:</u> $\begin{bmatrix} (1-\lambda) & 1 \\ 1 & (1-\lambda) \end{bmatrix}$ not invertible when its determinant is zero:

$(1-\lambda)^2 - 1 = 0$ iff $(1 - 2\lambda + \lambda^2) - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$. Only <u>roots</u> of this polynomial are $\lambda_1 = 0$ and $\lambda_2 = 2$. Solve two lin. systems:

For $\lambda_1 = 0$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ and for $\lambda_2 = 2$: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$.