<u>Def</u>. For $A = [a_{ij}] \in \mathbb{F}_n^m$ let

$\text{Row}_i(A) = [a_{i1}\ a_{i2} \cdots a_{in}] \in \mathbb{F}_n$ for $1 \le i \le m$,

$$\text{Col}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{F}^m \text{ for } 1 \le j \le n.$$

<u>Def</u>. For V a vector space over $\mathbb{F}$ and $S \overset{\text{subset}}{\subseteq} V$, a <u>linear combination</u> from S is an element $\sum_{i=1}^m x_i s_i \in V$ for some $x_i \in \mathbb{F}$, $s_i \in S$. The set of all such elements (vectors) is the <u>span of S</u> denoted by $\langle S \rangle$. If $S = \emptyset$ is empty, $\langle S \rangle = \{\theta_V\}$.

Def. For vector space V over $\mathbb{F}$, a subset $W \subseteq V$ is called a <u>subspace</u> of V when W is itself a vector space over $\mathbb{F}$ under the same operations $+$ and $\cdot$. Notation: $W \leq V$.

Th: Let $W \subseteq V$ be a subset of vector space V over $\mathbb{F}$. Then $W \leq V$ is a <u>subspace</u> of V iff

① W is closed under $+$, that is, $\forall w_1, w_2 \in W$, $w_1 + w_2 \in W$,

② W is closed under $\cdot$, that is, $\forall w \in W$, $\forall \alpha \in \mathbb{F}$, $\alpha \cdot w \in W$,

③ $\theta_V \in W$.

Pf. These conditions are certainly necessary by the definition of vector space.

If ①, ②, ③ are true, then all the other axioms for vector space being true in V implies they are true in subset W. Most are "univerally quantified" statements true for all elements of V, so they are also true for all elements of W. The axiom ⑥ about additive inverse is not of this type so it needs to be checked using the following lemma.

<u>Lemma.</u> Suppose V is a vector space over $\mathbb{F}$. Then

① $0 \cdot v = \theta, \quad \forall v \in V,$

② $\alpha \cdot \theta = \theta, \quad \forall \alpha \in \mathbb{F},$

③ $-v = (-1) \cdot v, \quad \forall v \in V.$

Pf. ① Let $w = 0 \cdot v \in V$. We have $0+0=0$ in $\mathbb{F}$ so $w = 0\cdot v = (0+0)\cdot v = (0\cdot v)+(0\cdot v) = w+w$ by the axiom ⑧. Since $\exists(-w) \in V$ by axiom ⑥, using assoc. axiom ③, we have

$$\theta = w+(-w) = (w+w)+(-w) = w+(w+(-w))$$
$$= w+\theta = w \quad \text{so} \quad 0\cdot v = \theta.$$

② Let $w = \alpha\cdot\theta$. Since $\theta = \theta+\theta$ we have

$$w = \alpha \cdot \theta = \alpha\cdot(\theta+\theta) = (\alpha\cdot\theta)+(\alpha\cdot\theta) = w+w.$$

So, as above, get $w = \theta$.

③ By uniqueness of inverse in any group, $-v$ is the element of V s.t. $v+(-v) = \theta$.

$$v+(-1)\cdot v \underset{⑩}{=} 1\cdot v+(-1)\cdot v \underset{⑧}{=} (1+(-1))\cdot v = 0\cdot v = \theta. \quad \square$$

So coming back to finish the proof of the last Theorem, axiom ⑥ holds in W because we are given that $\forall \alpha \in \mathbb{F}, \forall w \in W, \alpha \cdot w \in W$. So $-1 \cdot w = -w \in W$ by last Lemma, part ③. $\square$ 18

Prop: For $S \subseteq V$ any subset of v.s. V we have $\langle S \rangle \leq V$, the span of S is a subspace.

Pf. True by definition when $S = \emptyset$ is the empty set since $\langle S \rangle := \{\theta_V\}$ satisfies ①, ②, ③ of last Theorem:

$\theta + \theta = \theta \in \langle S \rangle$, $\alpha \cdot \theta = \theta \in \langle S \rangle$, $\forall \alpha \in \mathbb{F}$, $\theta \in \langle S \rangle$. So suppose S is not empty.

For $w_1, w_2 \in \langle S \rangle$, write
$w_1 = \sum_{i=1}^{m} x_i s_i$ and $w_2 = \sum_{j=1}^{n} y_j s_j'$ for some
$x_i, y_j \in \mathbb{F}$, $s_i, s_j' \in S$. Then
$w_1 + w_2 = \sum_{i=1}^{m} x_i s_i + \sum_{j=1}^{n} y_j s_j' \in \langle S \rangle$ is a
(finite) lin. comb. from S, so $\langle S \rangle$ is closed under $+$.

$\forall \alpha \in \mathbb{F}$, $w_1 \in \langle S \rangle$ as above, we have
$\alpha \cdot w_1 = \alpha \cdot \sum_{i=1}^{m} x_i \cdot s_i = \sum_{i=1}^{m} \alpha \cdot (x_i \cdot s_i) = \sum_{i=1}^{m} (\alpha \cdot x_i) \cdot s_i$
$\in \langle S \rangle$ so $\langle S \rangle$ is closed under $\cdot$.

$S \subseteq \langle S \rangle$ so for any $s_1 \in S$, $\theta = 0 \cdot s_1 \in \langle S \rangle$.
By last Theorem, $\langle S \rangle \leq V$. $\square$

Linear Systems: Let $1 \le m, n \in \mathbb{Z}$.

Def. A linear system of m equations in n variables is a list of the form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

where $a_{ij} \in \mathbb{F}$, $b_i \in \mathbb{F}$ for $1 \le i \le m$, $1 \le j \le n$, and x_j are variables whose values in $\mathbb{F}$ we are seeking such that all m equations are true simultaneously.

Given all "coefficients" a_{ij} and all "constant targets" b_i, the values x_j for which all equations are true are called the "solutions." If there are no solutions, we say the system is "<u>inconsistent</u>". If there is at least one solution, the system is called "<u>consistent</u>" and we write the <u>solution set</u> as

$$\left\{ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n \right\}$$ a subset of $\mathbb{F}^n$.

Let $A = [a_{ij}] \in \mathbb{F}_n^m$ be called the "<u>coefficient matrix</u>" of the lin. sys. and let

$B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{F}^m$ be the "constant matrix" of the lin. sys.

<u>Define</u> $AX = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} X_j \\ \sum_{j=1}^{n} a_{2j} X_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} X_j \end{bmatrix} = \sum_{j=1}^{n} X_j \, Col_j(A)$

as the column matrix made from the left hand sides of the m equations.

$\in \mathbb{F}^m$ is a lin. comb. of the columns of A.

Then the lin. sys. is a single <u>matrix</u> equation

$AX = B$ and the solution set is

$\{X \in \mathbb{F}^n \mid AX = B\}$.

Def. We say lin. sys. $AX=B$ is homogeneous when $B=0_1^m$, that is, all $b_i=0$, $1\le i\le m$. If any $b_i\ne 0$ we call the system inhomogeneous.

Note $AX=0$ is always consistent since it has the "trivial" solution $X=0_1^n$.
So the main question for $AX=0$ is whether it has any "non-trivial" solutions $X\ne 0_1^n$.

Th. For any $A=[a_{ij}]\in \mathbb{F}_n^m$, the solution set $W=\{X\in\mathbb{F}^n \mid AX=0_1^m\}$ of the homog. sys. $AX=0$ is a subspace of $\mathbb{F}^n$.

To prove this Theorem we first prove the next one.

Th: For any $A=[a_{ij}]\in \mathbb{F}^m_n$, $X, Y\in \mathbb{F}^n_1$, $\alpha\in \mathbb{F}$,

we have:

① $A(X+Y)=(AX)+(AY)$ ③ $A0^n_1=0^m_1$

② $A(\alpha\cdot X)=\alpha\cdot(AX)$

Pf ①

$$A(X+Y)=\begin{bmatrix}\sum_{j=1}^n a_{1j}(x_j+y_j)\\ \vdots\\ \sum_{j=1}^n a_{mj}(x_j+y_j)\end{bmatrix}=\begin{bmatrix}\sum_{j=1}^n (a_{1j}x_j+a_{1j}y_j)\\ \vdots\\ \sum_{j=1}^n (a_{mj}x_j+a_{mj}y_j)\end{bmatrix}$$

$$=\begin{bmatrix}\sum_{j=1}^n a_{1j}x_j\\ \vdots\\ \sum_{j=1}^n a_{mj}x_j\end{bmatrix}+\begin{bmatrix}\sum_{j=1}^n a_{1j}y_j\\ \vdots\\ \sum_{j=1}^n a_{mj}y_j\end{bmatrix}=AX+AY.$$

what algebra laws were used in each step?

② $$A(\alpha \cdot X) = \begin{bmatrix} \sum a_{1j}(\alpha X_j) \\ \vdots \\ \sum a_{mj}(\alpha X_j) \end{bmatrix} = \begin{bmatrix} \sum (a_{1j}\alpha) X_j \\ \vdots \\ \sum (a_{mj}\alpha) X_j \end{bmatrix}$$

$$= \begin{bmatrix} \sum (\alpha a_{1j}) X_j \\ \vdots \\ \sum (\alpha a_{mj}) X_j \end{bmatrix} = \begin{bmatrix} \sum \alpha (a_{1j} X_j) \\ \vdots \\ \sum \alpha (a_{mj} X_j) \end{bmatrix} = \begin{bmatrix} \alpha \sum a_{1j} X_j \\ \vdots \\ \alpha \sum a_{mj} X_j \end{bmatrix}$$

$$= \alpha \begin{bmatrix} \sum a_{1j} X_j \\ \vdots \\ \sum a_{mj} X_j \end{bmatrix} = \alpha (AX).$$

③ $$A O_1^n = \begin{bmatrix} \sum a_{1j} \cdot 0 \\ \vdots \\ \sum a_{mj} \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = O_1^m . \quad \square$$

Pf of $W = \{X \in \mathbb{F}^n \mid AX = 0^m_1\} \leq \mathbb{F}^n$:

① W is closed under $+$ because if $X, Y \in W$, $AX = 0^m_1$ and $AY = 0^m_1$ so $A(X+Y) = AX + AY = 0^m_1 + 0^m_1 = 0^m_1$ so $X+Y \in W$.

② W is closed under $\cdot$ because if $X \in W$, $\alpha \in \mathbb{F}$, $AX = 0^m_1$ so $A(\alpha \cdot X) = \alpha \cdot (AX) = \alpha \cdot 0^m_1 = 0^m_1$ so $\alpha \cdot X \in W$.

③ $0^n_1 \in W$ since $A0^n_1 = 0^m_1$. $\square$

Goal: Find efficient algorithm to solve linear systems.

Solution: Row reduction to Reduced Row Echelon Form (RREF) and interpretation.

Functional Viewpoint:

For $A = [a_{ij}] \in \mathbb{F}^m_n$ define the associated map

$$L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m \text{ by } L_A(X) = AX, \quad \forall X \in \mathbb{F}^n.$$

domain codomain

Th: The map L_A satisfies

① $L_A(X+Y) = L_A(X) + L_A(Y)$

② $L_A(\alpha \cdot X) = \alpha \cdot L_A(X)$

Pf. ① $L_A(X+Y) = A(X+Y) = AX + AY = L_A(X) + L_A(Y)$

② $L_A(\alpha \cdot X) = A(\alpha \cdot X) = \alpha \cdot (AX) = \alpha \cdot L_A(X)$.

These are just restatements of the last Theorem.

$\square$

Def. Let V and W be any vector spaces over $\mathbb{F}$. We say that a map (function) $L: V \to W$ is <u>linear</u> when

① $L(v_1+v_2) = L(v_1)+L(v_2)$, $\forall v_1, v_2 \in V$,

② $L(\alpha \cdot v) = \alpha \cdot L(v)$, $\forall v \in V$, $\forall \alpha \in \mathbb{F}$.

<u>Th</u> $L_A: \mathbb{F}^n \to \mathbb{F}^m$ defined by $L_A(X) = AX$ is linear.

Important questions about lin. maps:

① When is $L: V \to W$ <u>injective</u>? <u>surjective</u>? <u>bijective</u>? <u>invertible</u>?

These concepts come from the general Theory of functions between sets $f: S \to T$, so we will review them. But first:

Def. For $L: V \to W$ linear, define
$\mathrm{Ker}(L) = \{v \in V \mid L(v) = \theta_W\}$ and
$\mathrm{Range}(L) = \mathrm{Image}(L) = \{L(v) \in W \mid v \in V\}$.

Th: For $L: V \to W$ linear, we have
$\mathrm{Ker}(L) \leq V$ and $\mathrm{Range}(L) \leq W$ (subspaces).

Pf. If $v_1, v_2 \in \mathrm{Ker}(L)$ then $v_1 + v_2 \in \mathrm{Ker}(L)$
because $L(v_1 + v_2) = L(v_1) + L(v_2) = \theta_W + \theta_W = \theta_W$.
If $v \in \mathrm{Ker}(L)$ and $\alpha \in \mathbb{F}$ then $\alpha \cdot v \in \mathrm{Ker}(L)$
because $L(\alpha \cdot v) = \alpha \cdot L(v) = \alpha \cdot \theta_W = \theta_W$.
$\theta_V \in \mathrm{Ker}(L)$ since $L(\theta_V) = L(\theta_V + \theta_V) =$
$L(\theta_V) + L(\theta_V)$ so for $w = L(\theta_V)$, $w = w + w$
which gives $w = \theta_W$. So $\mathrm{Ker}(L) \leq V$. $\blacksquare$

If $w_1, w_2 \in Range(L)$, $\exists v_1, v_2 \in V$ s.t. $L(v_1) = w_1$ and $L(v_2) = w_2$ so $L(v_1+v_2) = L(v_1) + L(v_2) = w_1 + w_2$ so $w_1 + w_2 \in Range(L)$.
If $w = L(v) \in Range(L)$ and $\alpha \in \mathbb{F}$ then $\alpha \cdot w = \alpha \cdot L(v) = L(\alpha \cdot v) \in Range(L)$.
$\theta_W = L(\theta_V) \in Range(L)$. So $Range(L) \leq W$. $\square$

Should have done the following Lemma first.

<u>Lemma</u>. Suppose $L: V \to W$ is linear. Then

① $L(\theta_V) = \theta_W$, ② $L(-v) = -L(v)$, $\forall v \in V$,

③ $L\left(\sum_{i=1}^{m} x_i v_i\right) = \sum_{i=1}^{m} x_i L(v_i)$, $\forall v_i \in V$, $x_i \in \mathbb{F}$, $1 \leq m$.