Def: For $L: V \to V$, $\lambda \in \mathbb{R}$, let
$L_\lambda = \{v \in V \mid L(v) = \lambda v\}$. Note: $L(\theta) = \theta = \lambda \cdot \theta$
so $\theta \in L_\lambda$.

Th: $L_\lambda \leq V$, called the λ eigenspace of
L if $L_\lambda \neq \{\theta\}$ non-trivial subspace
whose nonzero vectors are all e-vectors for
L with e-value λ.

Pf: $\theta \in L_\lambda$ done. If $u, v \in L_\lambda$ then
$L(u) = \lambda u$ and $L(v) = \lambda v$ so
$L(u+v) = L(u) + L(v) = \lambda u + \lambda v = \lambda(u+v)$ so
$u+v \in L_\lambda$.

If $u \in L_\lambda$ and $c \in \mathbb{R}$, $L(u) = \lambda u$ and $L(cu) = cL(u) = c(\lambda u) = (c\lambda)u = (\lambda c)u = \lambda(cu)$ so $cu \in L_\lambda$. $\square$

General Procedure: Given $L: V \to V$

① Find all $\lambda \in \mathbb{R}$ s.t. $L_\lambda \neq \{\theta\}$

② List them $\lambda_1, \lambda_2, \ldots, \lambda_r$ (distinct)

③ For each λ_i find a basis of L_{λ_i}, $T_i = \{w_{i1}, w_{i2}, \ldots, w_{ig_i}\}$, $g_i = \dim(L_{\lambda_i})$ called geometric multiplicity of λ_i for L.

④ Is $T = T_1 \cup T_2 \cup \cdots \cup T_r$ a basis for V ?

⑤ If T is a basis for V, it is an e-basis, L is diag-able, and

$$
{}_T[L]_T = \begin{bmatrix} \boxed{\lambda_1 I_{g_1}} & & & O \\ & \boxed{\lambda_2 I_{g_2}} & & \\ & & \ddots & \\ O & & & \boxed{\lambda_r I_{g_r}} \end{bmatrix} = D \text{ is a diagonal matrix}
$$

representing L w.r.t. T.

If $A = {}_S[L]_S$, $\boxed{P = {}_S P_T}$ = transition matrix from T to S

then $D = P^{-1} A P$ has "diagonalized" A.

Example: Try to diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^3_3$.

Step ①: Find all possible e-values $\lambda \in \mathbb{R}$ s.t. $AX = \lambda X$ for $0^3_1 \neq X \in \mathbb{R}^3$. $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Suppose $AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then it means

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda I_3 X$$

same as $(*)$ $(A - \lambda I_3) X = 0^3_1$ so looking for $\lambda \in \mathbb{R}$ s.t. $Nul(A - \lambda I_3)$ is <u>non-trivial</u>

$(*)$ has non-zero solutions.

$$[A-\lambda I_3 \mid 0^3_1] = \left[\begin{array}{ccc|c} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 1 & 1 & 1-\lambda & 0 \end{array}\right]$$

$$\begin{array}{ccc} -1 & -1 & \lambda-1 \end{array}$$

try row op's
$-R_3+R_1 \to R_1$
$-R_3+R_2 \to R_2$

get

$$\left[\begin{array}{ccc|c} -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & \lambda & 0 \\ 1 & 1 & 1-\lambda & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1-\lambda & 0 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & -\lambda & 0 \end{array}\right]$$

Two cases:
$\lambda = 0$ or
$\lambda \neq 0$

Case ① $\lambda = 0$:

$\boxed{\lambda_1 = 0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -r-s \\ x_2 = r \\ x_3 = s \end{array} \Big\} \text{free}$$

$$Nul(A-\lambda I_3) = Nul(A) = A_0 = \left\{ \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r,s \in \mathbb{R} \right\}$$

has basis $T_1 = \left\{ \underset{w_{11}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{w_{12}}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$ $\quad g_{\lambda_1} = \dim(A_{\lambda_1}) = 2$

case ②: $\lambda \neq 0$: $\left[\begin{array}{ccc|c} 1 & 1 & (1-\lambda) & 0 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & -\lambda & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & (1-\lambda) & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$\begin{array}{ccc} -1 & 0 & 1 \\ 0 & -1 & 1 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3-\lambda & 0 \end{array}\right]$ has free variables iff $\lambda = 3$. So only other e-value is $\boxed{\lambda_2 = 3}$

To get e-space $A_3 = Nul(A - 3I_3)$

solve $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} x_1 = r \\ x_2 = r \\ x_3 = r \text{ free} \end{array}$ $A_3 = \left\{ \begin{bmatrix} r \\ r \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$

has basis $T_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = w_{21} \right\}$, $g_{\lambda_2} = \dim(A_3) = 1$

Is $T = T_1 \cup T_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ an e-basis of $\mathbb{R}^3$ for A? If so, find $P = {}_S P_T$, $D = P^{-1} A P$.

Check: $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$\quad A \quad w_{11} \qquad\qquad = 0\, w_{11}$

$$\underset{A}{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}\underset{w_{12}}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\underset{=0\,w_{12}}{0\begin{bmatrix}-1\\0\\1\end{bmatrix}} \text{ and } \underset{A}{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}\underset{w_{21}}{\begin{bmatrix}1\\1\\1\end{bmatrix}}=\begin{bmatrix}3\\3\\3\end{bmatrix}=\underset{=3\,w_{21}}{3\begin{bmatrix}1\\1\\1\end{bmatrix}}$$

$\left[\begin{array}{ccc|ccc}1&0&0&-1&-1&1\\0&1&0&1&0&1\\0&0&1&0&1&1\end{array}\right]$ is in RREF so ${}_S P_T=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}=P$

$\quad S \quad T$ (as columns)

$$\underset{\substack{T\quad S\\ \text{(as columns)}}}{\left[\begin{array}{ccc|ccc}-1&-1&1&1&0&0\\1&0&1&0&1&0\\0&1&1&0&0&1\end{array}\right]}\to\left[\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&3&1&1&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&-1/3&2/3&-1/3\\0&1&0&-1/3&-1/3&2/3\\0&0&1&1/3&1/3&1/3\end{array}\right]$$

$0 \quad 0 \quad -1 \quad -1/3 \quad -1/3 \quad -1/3$

$$\frac{1}{3}\begin{bmatrix}-1&2&-1\\-1&-1&2\\1&1&1\end{bmatrix} \quad {}_T P_S = \\ = P^{-1}$$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$= D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$ is diagonal, $\boxed{\begin{matrix}\lambda_1 = 0 \\ \lambda_2 = 3\end{matrix}}$

Note: $\lambda_1 = 0$ was repeated corresponding to $g_{\lambda_1} = 2$.

<u>Exercise</u>: Show that $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ can <u>not</u> be diagonalized using only real numbers.

<u>Th</u>: Let $L: V \to V$ have <u>distinct</u> e-values $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ with corresponding e-vectors $w_1, \dots, w_r \in V$, so $L(w_i) = \lambda_i w_i$ for $1 \le i \le r$. Then $S = \{w_1, \dots, w_r\} \subseteq V$ is <u>independent</u>.

<u>Proof.</u> Suppose $\theta = \sum_{i=1}^{m} c_i w_i$ is a <u>shortest</u> dep. relation on S, so all $c_i \neq 0$, $1 \le i \le m$. We may have relabeled vectors for convenience. Apply L to get $\theta = \sum_{i=1}^{m} c_i L(w_i) = \sum_{i=1}^{m} c_i \lambda_i w_i$

Could also just multiply the dep rel. by λ_1, get $\theta = \sum_{i=1}^{m} c_i \lambda_1 w_i$. Subtract 2^{nd} eq. from 1^{st}:

Get $\theta = \sum_{i=1}^{m} c_i(\lambda_i - \lambda_1) w_i = \sum_{i=2}^{m} c_i(\lambda_i - \lambda_1) w_i$

Note: $m \geq 2$ since $\theta = c_1 w_1$ can't happen
($c_1 \neq 0$, $w_1 \neq \theta$).

The $i=1$ term is $c_1(\lambda_1 - \lambda_1) w_1 = c_1 (0) w_1 = \theta$.
For $2 \leq i \leq m$, $\lambda_i - \lambda_1 \neq 0$ (distinct e-values),
so $\theta = \sum_{i=2}^{m} c_i (\lambda_i - \lambda_1) w_i$ is a <u>shorter dep. rel on S</u>, contradicting "shortest" chosen. $\square$

<u>Th</u>: Let $L: V \to V$ have distinct e-values $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ and for $1 \leq i \leq r$ let $T_i = \{w_{i1}, \dots, w_{ig_i}\}$ be a basis of e-space L_{λ_i}.
Then $T = T_1 \cup T_2 \cup \dots \cup T_r$ is <u>independent</u>.

Note: $T = \{w_{11}, \dots, w_{1g_1}, w_{21}, \dots, w_{2g_2}, \dots, w_{r1}, \dots, w_{rg_r}\}$ is a list of $g_1 + g_2 + \dots + g_r$ e-vectors in V.

Proof. Since $w_{ij} \in T_i \subseteq L_{\lambda_i}$ for $1 \le i \le r$ and $1 \le j \le g_i$ we have $L(w_{ij}) = \lambda_i w_{ij}$.

Suppose T were dependent, so there is a "shortest" dep. rel. on T. If necessary, by relabeling sets T_i and vectors in T_i, we could write that dep. rel. as

$$\theta = \sum_{i=1}^{m} \sum_{j=1}^{h_i} c_{ij} w_{ij} \quad \text{where each } c_{ij} \neq 0.$$

$$= \sum_{i=1}^{m} w_i \quad \text{where } w_i = \sum_{j=1}^{h_i} c_{ij} w_{ij} \in L_{\lambda_i}$$

and each $w_i \neq \theta$ since T_i is indep and $c_{ij} \neq 0$.

But that contradicts the last Theorem, so T indep. $\square$

This means T is a basis of V iff $g_1+g_2+\cdots+g_r = n = \dim(V)$, so L is diag-able iff we get a basis of e-vectors for V, enough from each e-space L_{λ_i} to make an e-basis for all of $V=\langle T\rangle$.

Focus now on finding all distinct e-values $\lambda_1, \dots, \lambda_r$ for $L: V \to V$ or for $A \in \mathbb{R}^n_n$.

Find all $\lambda \in \mathbb{R}$ s.t. $(L-\lambda I_V)(v)=\theta$ has solutions $v \neq \theta$, that is, s.t. $\ker(L-\lambda I_V) \neq \{\theta_V\}$. Let S be any basis of V, and $A = {}_S[L]_S$ so $A - \lambda I_n = {}_S[L-\lambda I_V]_S$.

$\mathrm{Nul}(A-\lambda I_n) \neq \{\theta_1^n\}$ iff $\mathrm{rank}(A-\lambda I_n) < n$
iff $A-\lambda I_n$ is <u>not invertible</u>.
Use determinants to study this, and for other uses.

<u>Def.</u> For $1 \leq n \in \mathbb{Z}$ (integer) let
$S = \{1, 2, \ldots, n\}$ and let
$S_n = \mathrm{Perm}(S) = \{f : S \to S \mid f \text{ is bijective}\}$

Notation: Write $f = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ f(1) & f(2) & \cdots & f(i) & \cdots & f(n) \end{pmatrix}$
like a table of values. <u>Examples:</u>

$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$ has only two elements.

$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$ has six elements.

For $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$ there are

# of choices: $n\,(n-1)\cdots(2)(1) = n!$ ("n factorial")

So S_n contains $n!$ distinct elements, each one a bijection from S to S.

Composition of any two elements of S_n is another one, so have a <u>binary operation</u> on S_n, $\circ$, composition.

<u>Example</u>: For $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

find $f \circ g$ and $g \circ f$.

$$\begin{aligned} g &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ f &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ \hline f \circ g &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} f &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ g &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ \hline g \circ f &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{aligned}$$

Note: $f \circ g \neq g \circ f$ can happen.

<u>Note</u>: $I = I_S = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ 1 & 2 & \cdots & i & \cdots & n \end{pmatrix}$ the identity map on S is bijective so $I \in S_n$ and $\forall f \in S_n$, $f \circ I = f = I \circ f$, so have an identity element for $\circ$ in S_n.

<u>Also</u>: $\forall f \in S_n$, $f^{-1} \in S_n$ since bijections are invertible and their inverses are bijective.

Ex: $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$ has $f^{-1} = \begin{pmatrix} 2 & 5 & 4 & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

$\left(\text{just permute columns to get top row in order}\right) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$

<u>Th</u>: $(S_n, \circ, I)$ is a group under composition with id. elt. I.