Def: For $f = \begin{pmatrix} 1 & \cdots & i & \cdots & j & \cdots & n \\ f(1) & & f(i) & & f(j) & & f(n) \end{pmatrix} \in S_n$

Say f has an <u>inversion</u> for the pair (i, j), $1 \leq i < j \leq n$, when $f(i) > f(j)$.

EX: Inversions of $f = \begin{pmatrix} 1 & 2 & 3 \\ \underbrace{2 & 1} & 3 \end{pmatrix} \in S_3$ (only one)

are marked below:

$f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ has 3 inversions

Def: Let $\mathrm{Inv}(f)$ = Total number of inversions in $f \in S_n$.

Def: For $f \in S_n$ let $\mathrm{sgn}(f) = (-1)^{\mathrm{Inv}(f)}$ $\in \{1, -1\}$ (say f is even or odd)

Def: For $A = [a_{ij}] \in \mathbb{R}^n_n$ define

$$|A| = \det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

a sum ($n!$ terms) one term for each $\sigma \in S_n$

each term a product of n entries from A, one from each row, column number depends on σ.

Ex: $n = 2$: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$

$\operatorname{sgn}(\sigma)$: $\quad 1 \quad -1$

$$\det(A) = (+1) a_{11} a_{22} + (-1) a_{12} a_{21}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$ is usual 2×2 $\det(A)$ formula.

"Crosshatching"

Ex: $n = 3$: $A = [a_{ij}] \in \mathbb{R}^3_3$

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

sgn(σ): 1, -1, -1, $+1$, $+1$, -1

$$\det(A) = 1\, a_{11} a_{22} a_{33} + 1\, a_{12} a_{23} a_{31} + 1\, a_{13} a_{21} a_{32}$$
$$- 1\, a_{11} a_{23} a_{32} - 1\, a_{12} a_{21} a_{33} - 1\, a_{13} a_{22} a_{31}$$

Crosshatching Method:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Warning: Crosshatching ONLY works for $n = 2, 3$

Products of three "+1" terms $\diagdown\diagdown\diagdown$

Products of three "−1" terms $\diagup\diagup\diagup$

Ex: $\det\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 0 & 4 & 5 \end{bmatrix}\begin{matrix} 1 & -1 \\ 3 & 1 \\ 0 & 4 \end{matrix}$

$= (1)(1)(5) + (-1)(-1)(0) + (2)(3)(4)$
$\quad - (2)(1)(0) - (1)(-1)(4) - (-1)(3)(5)$

$= 5 + 24 + 4 + 15 = 48$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{\text{adder}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{\text{adder}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 12 \end{bmatrix}$$

$-3 \quad 3 \quad -6 \qquad\qquad 0 \quad -4 \quad 7 \qquad\qquad B$

$\det(B) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 12 \end{bmatrix}\begin{matrix} 1 & -1 \\ 0 & 4 \\ 0 & 0 \end{matrix} = (1)(4)(12) = 48$

How do row operations affect $\det(A)$?

Facts about det(A).

Th: Let $A=[a_{ij}]\in\mathbb{R}^n_n$. Then we have

(a) If A has a row of zeros then $\det(A)=0$.

(b) $\det(A^T)=\det(A)$

(c) If A has two identical rows, then $\det A=0$

(d) If $\operatorname{rank}(A)<n$ then $\det(A)=0$

(e) $\det(A)=0$ implies $\operatorname{rank}(A)<n$

(f) $\det(A)=0$ iff A is <u>not</u> invertible

Goal: Understand how elementary row operations affect $\det(A)$.

<u>Exercise</u>: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ If $X \in \mathbb{R}^2$ were an e-vector for A with e-value $\lambda \in \mathbb{R}$, then $(A - \lambda I_2)X = 0^2_1$ would have <u>non-zero</u> solutions.

But $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ and

$$\left[\begin{array}{cc|c} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \lambda & 0 \\ 0 & \lambda^2+1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

since $\lambda^2 + 1 \geq 1 > 0$

so $AX = \lambda X$ has only solution $X = 0^2_1$

Alternatively, $\det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = (-\lambda)^2 - (-1)(1) = \lambda^2 + 1 \neq 0$

so $\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ is invertible for all $\lambda \in \mathbb{R}$.

This A cannot be diagonalized "over $\mathbb{R}$".

In textbook is a discussion of the complex numbers $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R},\ i^2 = -1\}$ a very important "<u>field</u>" containing $\mathbb{R}$ as well as "imaginary" numbers like $i = \sqrt{-1}$.
Linear algebra can be done over <u>any</u> field, using scalars from the field instead of from $\mathbb{R}$.
This topic is developed in Advanced Linear Algebra, Math 404.

Review: For $A = [a_{ij}] \in \mathbb{R}^n_n$, define

$$\det(A) = |A| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where $\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(i) & \cdots & \sigma(j) & \cdots & \sigma(n) \end{pmatrix} \in S_n$

is any bijection $\sigma : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$.

$\operatorname{sgn}(\sigma) = (-1)^{\operatorname{Inv}(\sigma)} \in \{\pm 1\}$ where

$\operatorname{Inv}(\sigma) = \#$ inversions in σ

An inversion in σ is a pair (i, j) such that

$1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$.

Note: There are $n! = n(n-1)\cdots(2)(1)$ distinct bijections in S_n. Also called "permutations" since the values of σ are the numbers $1, \dots, n$ in some order.

Th: If $A=[a_{ij}]$ has a zero row then $\det(A)=0$.

Pf: If row r of A is all zeros, then $a_{rj}=0$ for all $1\le j\le n$, so in the formula for $\det(A)$ every term has a factor $a_{r\sigma(r)}=0$, so every term is 0. $\square$

Th: If $A=[a_{ij}]$ is upper triangular, so if $i>j$ then $a_{ij}=0$, then $\det(A)=a_{11}a_{22}\cdots a_{nn}$

Pf: Let $\sigma\in S_n$. Either $\sigma(n)=n$ or $\sigma(n)<n$. If $\sigma(n)<n$ then $a_{n\sigma(n)}=0$ so those terms contribute nothing to $\det(A)$. Consider remaining $\sigma\in S_n$ s.t. $\sigma(n)=n$, so

$\sigma = \begin{pmatrix} 1 & 2 & \cdots & (n-1) & n \\ \sigma(1) & \sigma(2) & & \sigma(n-1) & n \end{pmatrix}$. Either

$\sigma(n-1) = n-1$ or $\sigma(n-1) < n-1$.

If $\sigma(n-1) < n-1$ then $a_{(n-1)\sigma(n-1)} = 0$ $(i>j)$

so those terms contribute nothing to $|A|$.

Consider remaining $\sigma \in S_n$ s.t.

$\sigma(n-1) = n-1$ and $\sigma(n) = n$. So

$\sigma = \begin{pmatrix} 1 & 2 & \cdots & (n-2) & (n-1) & n \\ \sigma(1) & \sigma(2) & & \sigma(n-2) & (n-1) & n \end{pmatrix}$. Either

$\sigma(n-2) = n-2$ or $\sigma(n-2) < n-2$.

As before, $\sigma(n-2) < n-2$ gives $a_{(n-2)\sigma(n-2)} = 0$

so get no contributions to $|A|$. Can

consider only $\sigma \in S_n$ s.t. $\sigma(n-2) = n-2$, $\sigma(n-1) = n-1$

and $\sigma(n) = n$. Continue same argument, get

only contribution to $\det(A)$ is from $\sigma = I$,

$\sigma = I = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ 1 & 2 & \cdots & i & \cdots & n \end{pmatrix}$ and $\mathrm{sgn}(I) = 1$

so $\det(A) = a_{11} a_{22} \cdots a_{ii} \cdots a_{nn}$. $\square$

<u>Th</u>: Let $A = [a_{ij}] \in \mathbb{R}^n_n$ and suppose $B = [b_{ij}]$ is obtained from A by doing an elementary row operation to A. Then we have:

(1) $\det(B) = -\det(A)$ if row op. is a <u>switcher</u>,

(2) $\det(B) = c \det(A)$ if row op. is <u>multip. by c</u>

(3) $\det(B) = \det(A)$ if row op. is an <u>adder</u>.

<u>Proof</u>: (2) is easiest from definition of $|A|$.

Suppose B is obtained from A by multiplying row r of A by $c \in \mathbb{R}$ (even if $c = 0$).

Then $b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq r \\ c a_{ij} & \text{if } i = r \end{cases}$ so

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)} \cdots b_{r\sigma(r)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots c\, a_{r\sigma(r)} \cdots a_{n\sigma(n)}$$

$$= c \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{n\sigma(n)}$$

$$= c \det(A).$$

Before doing proof of (1), need fact about sgn.

<u>Th</u>: For any $\sigma, \tau \in S_n$ we have

$$\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau)$$

Group Theory.

__Example__: For $\sigma = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ and $\tau = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$

$$\operatorname{sgn}(\sigma) = (-1)^2 = 1 \quad \text{and} \quad \operatorname{sgn}(\tau) = (-1)^1 = -1$$

$$\tau = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$$

$$\sigma = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$$

$$\sigma\circ\tau = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$$

$$\operatorname{sgn}(\sigma\circ\tau) = (-1)^3 = -1$$

$$= \operatorname{sgn}(\sigma)\cdot\operatorname{sgn}(\tau)$$

$$= (1)\cdot(-1).$$

For $1 \le r < s \le n$ let $\tau \in S_n$ be the permutation that __just__ switches r and s:

$\tau = \begin{pmatrix}1&2&\cdots&r&\cdots&s&\cdots&n\\1&2&\cdots&s&\cdots&r&\cdots&n\end{pmatrix}$. Then for any

$\sigma = \begin{pmatrix}1&2&\cdots&r&\cdots&s&\cdots&n\\\sigma(1)&\sigma(2)&\cdots&\sigma(r)&\cdots&\sigma(s)&\cdots&\sigma(n)\end{pmatrix}$ we have

$$\sigma\circ\tau = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & s & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(s) & \cdots & \sigma(r) & \cdots & \sigma(n) \end{pmatrix}.$$

<u>Th</u>: $\operatorname{sgn}(\tau) = -1$. <u>Pf</u> Count Inversions.

<u>Pf</u> (continued) (1) Suppose $1 \le r < s \le n$ and B is obtained from A by switching rows r and s. Then

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \ne r \text{ and } i \ne s \\ a_{sj} & \text{if } i = r \\ a_{rj} & \text{if } i = s \end{cases}$$

so

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)} \cdots b_{r\sigma(r)} \cdots b_{s\sigma(s)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots \underbrace{a_{s\sigma(r)} \cdots a_{r\sigma(s)}}_{\text{out of order}} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(s)} \cdots a_{s\sigma(r)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1(\sigma\circ\tau)(1)} \cdots a_{r(\sigma\circ\tau)(r)} \cdots a_{s(\sigma\circ\tau)(s)} \cdots a_{n(\sigma\circ\tau)(n)}$$

As σ varies over all elements of S_n, so does $\sigma\circ\tau$ because the function $f_\tau : S_n \to S_n$ defined by $f_\tau(\sigma) = \sigma\circ\tau$ is <u>bijective!</u> (Exercise in group theory.)

So, let $\mu = \sigma\circ\tau = f_\tau(\sigma)$ be a new index:

$$\det(B) = \sum_{\mu \in S_n} \operatorname{sgn}(\sigma)\, a_{1\mu(1)} \cdots a_{n\mu(n)}$$

$$= -\sum_{\mu \in S_n} \operatorname{sgn}(\mu)\, a_{1\mu(1)} \cdots a_{n\mu(n)} = -|A|.$$

$\operatorname{sgn}(\mu) = \operatorname{sgn}(\sigma\circ\tau) = \operatorname{sgn}(\sigma)\cdot\operatorname{sgn}(\tau) = -\operatorname{sgn}(\sigma)$

Corollary of (1): If A has two identical rows then $|A|=0$.

Pf: If B is obtained from A by switching those two identical rows, then $B=A$ and $\det(B)=-\det(A)$ so $\det(A)=-\det(A)$ so $2\det(A)=0$ so $\det(A)=0$.

Pf of (3): Suppose B is obtained from A by elementary adder row operation $cR_r+R_s \to R_s$ and then

so $b_{ij}=\begin{cases} a_{ij} & \text{if } i\neq s \\ a_{sj}+ca_{rj} & \text{if } i=s \end{cases}$

$$|B|=\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, \underset{\substack{\parallel \\ a_{1\sigma(1)}}}{b_{1\sigma(1)}} \cdots \underset{\substack{\parallel \\ a_{r\sigma(r)}}}{b_{r\sigma(r)}} \cdots \underset{\substack{\parallel \\ (a_{s\sigma(s)}+ca_{r\sigma(s)})}}{b_{s\sigma(s)}} \cdots \underset{\substack{\parallel \\ a_{n\sigma(n)}}}{b_{n\sigma(n)}}$$

$$|B| = \sum_{\sigma \in S_n} \text{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{s\sigma(s)} \cdots a_{n\sigma(n)}$$

$$+ c \sum_{\sigma \in S_n} \text{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{r\sigma(s)} \cdots a_{n\sigma(n)}$$

The first sum is just $|A|$ so why is the second sum zero?

Let $D = [d_{ij}]$ be the matrix obtained from A by <u>replacing</u> row s by row r, that is,

$$d_{ij} = \begin{cases} a_{ij} & \text{if } i \neq s \\ a_{rj} & \text{if } i = s. \end{cases}$$

Then

$$|D| = \sum_{\sigma \in S_n} \text{sgn}(\sigma)\, d_{1\sigma(1)} \cdots d_{r\sigma(r)} \cdots d_{s\sigma(s)} \cdots d_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{r\sigma(r)} \cdots a_{r\sigma(s)} \cdots a_{n\sigma(n)}$$

is that second sum above. $\square$

How do we use these Theorems to efficiently calculate det(A)?

Ex:

$$\begin{vmatrix}1&1&-1&-1\\2&3&4&1\\3&2&1&1\\4&4&-2&-3\end{vmatrix} = \begin{vmatrix}1&1&-1&-1\\0&1&6&3\\0&-1&4&4\\0&0&2&1\end{vmatrix} = \begin{vmatrix}1&1&-1&-1\\0&1&6&3\\0&0&10&7\\0&0&2&1\end{vmatrix} =$$

$-2 \ -2 \ \ 2 \ \ 2$

$-3 \ -3 \ \ 3 \ \ 3$

$-4 \ -4 \ \ 4 \ \ 4$

$0 \ \ 0 \ -10 \ -5$

switch

$$\begin{vmatrix}1&1&-1&-1\\0&1&6&3\\0&0&0&2\\0&0&2&1\end{vmatrix} = -\begin{vmatrix}1&1&-1&-1\\0&1&6&3\\0&0&2&1\\0&0&0&2\end{vmatrix} = -4$$

Crosshatching was **NOT** an option!!
The definition would have involved adding 24 terms, not efficient or reasonable.

Ex: $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 6 & 6 & 9 \end{vmatrix} = (2)(3) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 6$

is faster than crosshatching method for 3x3.

Th: Suppose E is an elementary matrix associated with an elem. row operation. Then

(1) $\det(E) = -1$ if E is a switcher,

(2) $\det(E) = c$ if E is a multiplier by c.

(3) $\det(E) = 1$ if E is an adder.

Pf: In each case, E is obtained from I_n by doing the row op. to I_n, and $\det(I_n) = 1$, so these follow from the theorem giving the effect on det of doing elem. row ops. $\square$