Cor: Let E be the elem. matrix associated with an elem. row op, so that EA is the matrix obtained from A by doing that row op. to A. Then $\det(EA) = (\det E)(\det A)$.

Pf: This is the result of the last theorems.

Th: Suppose B is obtained from A by a sequence of elem. row ops corresponding to elem. matrices $E_1, E_2, \dots, E_r$. Then

$B = E_r \cdots E_2 E_1 A$ and

$\det(B) = \det(E_r)\det(E_{r-1}) \cdots \det(E_2)\det(E_1)\det A$

and for each $1 \le i \le r$, $\det(E_i) \neq 0$.

Pf: Follows from last theorem.

Th: A is invertible iff $\det(A) \neq 0$.

Pf: A is invertible iff A row reduces to I_n iff $I_n = E_r \cdots E_2 E_1 A$ for some elem. matrices $E_1, E_2, \ldots, E_r$, so

$$1 = \det(I_n) = (\det E_r) \cdots (\det E_1)(\det A)$$

and each $\det(E_i) \neq 0$ so A invertible implies $\det(A) \neq 0$.

If A is not invertible, it row reduces to a matrix C with a zero row, $C = E_r \cdots E_1 A$

$$0 = \det C = (\det E_r) \cdots (\det E_1)(\det A)$$

and each $\det(E_i) \neq 0$ so $\det(A) = 0$. □

We only left out the proof that $\det(A^T) = \det(A)$.

Application of det to finding e-values:

Th: λ is an e-value of $A \in \mathbb{R}^n_n$ iff $A - \lambda I_n$ is not invertible iff $\det(A - \lambda I_n) = 0$.

Ex:
$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & \lambda \\ 0 & -\lambda & \lambda \\ 1 & 1 & (1-\lambda) \end{vmatrix} = \lambda^2 \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix}$$

$$-1 \quad -1 \quad \lambda-1$$

$$= \lambda^2 \begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & (3-\lambda) \end{vmatrix} = \lambda^2(3-\lambda)$$ is a polynomial of degree $n=3$ whose roots are the e-values of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Note: $-\lambda^2(\lambda-3)$ is <u>factored</u> into 3 linear factors: $-(\lambda-0)(\lambda-0)(\lambda-3) = -(\lambda-\lambda_1)^2(\lambda-\lambda_2)^1$

Th: For any $A, B \in \mathbb{R}^n_n$, $\det(AB) = (\det A)(\det B)$ 188

Pf: If A is invertible, $A = E_r \cdots E_2 E_1$ is a product of elem. matrices, so

$\det(AB) = \det(E_r \cdots E_1 B) = (\det E_r) \cdots (\det E_1)(\det B)$

$= \det(E_r \cdots E_1)(\det B) = (\det A)(\det B)$.

Suppose A is not invertible so $C = E_r \cdots E_1 A$ where C has a zero row, so $CB = E_r \cdots E_1 AB$ has a zero row so $\det(CB) = 0 = \det(E_r \cdots E_1 AB)$

$= (\det E_r) \cdots (\det E_1) \det(AB)$ so $\det(AB) = 0$

$= (\det A)(\det B)$ since $\det(A) = 0$. $\square$

Another method to compute det(A):

<u>Cofactor Expansion:</u>

<u>Def</u>: For $A=[a_{ij}]\in\mathbb{R}_n^n$, $1\le r, s\le n$, let $M_{rs}\in\mathbb{R}_{n-1}^{n-1}$ be the matrix obtained from A by deleting row r and column s.

<u>Ex:</u> For $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$, $M_{11}=[a_{22}]$, $M_{12}=[a_{21}]$

$M_{21}=[a_{12}]$, $M_{22}=[a_{11}]$

For $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, $M_{11}=\begin{bmatrix}5&6\\8&9\end{bmatrix}$, $M_{12}=\begin{bmatrix}4&6\\7&9\end{bmatrix}$

$M_{13}=\begin{bmatrix}4&5\\7&8\end{bmatrix}$, $M_{21}=\begin{bmatrix}2&3\\8&9\end{bmatrix}$, $M_{22}=\begin{bmatrix}1&3\\7&9\end{bmatrix}$, etc.

Def: With notation as above, let

$$A_{rs} = (-1)^{r+s}|M_{rs}| = (-1)^{r+s}\det(M_{rs}).$$

Th (Cofactor Expansion) For each $1 \le r, s \le n$, wehdre

(a) $\det(A) = \sum_{j=1}^{n} a_{rj} A_{rj}$ (expansion along row r)

(b) $\det(A) = \sum_{i=1}^{n} a_{is} A_{is}$ (expansion along column s)

Ex: $n=2$: $|A| = a_{11}A_{11} + a_{12}A_{12}$

$= a_{11}(-1)^{1+1}|M_{11}| + a_{12}(-1)^{1+2}|M_{12}|$

$= a_{11}a_{22} - a_{12}a_{21}$

row 1 cofactor expansion.

$$|A| = a_{21}A_{21} + a_{22}A_{22} \quad \text{(row 2 expansion)}$$
$$= a_{21}(-1)^{2+1}|M_{21}| + a_{22}(-1)^{2+2}|M_{22}|$$
$$= -a_{21}a_{12} + a_{22}a_{11}$$

$$|A| = a_{11}A_{11} + a_{21}A_{21} \quad \text{(column 1 expansion)}$$
$$= a_{11}(-1)^{1+1}|M_{11}| + a_{21}(-1)^{2+1}|M_{21}|$$
$$= a_{11}a_{22} - a_{21}a_{12}$$

<u>Ex</u>: $n=3$: $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= a_{11}(-1)^{1+1}|M_{11}| + a_{12}(-1)^{1+2}|M_{12}| + a_{13}(-1)^{1+3}|M_{13}|$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Value of cofactor expansion is best when used on a row (or column) with most number of 0 entries.

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 8 & 9 \end{vmatrix} = 5(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 5(9-21) = -60$

(using cofactor expansion along row 2)

$$\begin{vmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & 3 & 4 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{vmatrix} = 2(-1)^{3+2}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -10$$

(cof. exp along col. 2) (row ops.)

shows efficient use of a combination of methods.

Recall: $\det(AB) = (\det A)(\det B)$.

<u>Cor</u>: If A is invertible then
$$\det(A^{-1}) = \frac{1}{\det(A)} = (\det A)^{-1}.$$

<u>Pf</u>: $I_n = AA^{-1}$ so $1 = \det(I_n) = (\det A)(\det(A^{-1}))$
so $\det(A^{-1}) = \dfrac{1}{(\det A)}$. $\square$

<u>Application</u>: If $|A| = 5$, $|B| = 6$, $|C| = 7$
then $|A^2 B^T C^{-1}| = \dfrac{|A|^2 |B|}{|C|} = \dfrac{(25)(6)}{7} = \dfrac{150}{7}$
since $|A^2| = |A \cdot A| = |A| \cdot |A| = |A|^2$ and $|B^T| = |B|$.
(transpose)

Th: For $A \in \mathbb{R}^n_n$, $|A-\lambda I_n| = (-1)^n |\lambda I_n - A|$ is a polynomial in λ of degree n whose highest term is $(-1)^n \lambda^n$ and whose lowest constant term is $|A|$.

Pf: By definition of $|A-\lambda I_n|$, it is a sum of products of the entries of $A-\lambda I_n = [a_{ij} - \lambda \delta_{ij}]$ so each factor in each term is either a constant a_{ij} or a linear polynomial $a_{ii}-\lambda$. The sum of such products is a polynomial in λ with top term coming from $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$.

$\square$

Def: The *characteristic* polynomial of A is $|\lambda I_n - A| = \lambda^n + \cdots + (-1)^n |A|$.

Def: If char. poly. of A factors as

$$|\lambda I_n - A| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{k_i} = (\lambda-\lambda_1)^{k_1} \cdots (\lambda - \lambda_r)^{k_r}$$

for *distinct* roots $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, the powers $k_1, \dots, k_r$ are called the *algebraic multiplicities* of those roots, which are the *e-values* of A, say

k_i = algeb. mult. of λ_i for A.

Note: $n = k_1 + k_2 + \cdots + k_r$

Ex: We did $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and found

$|A - \lambda I_3| = -\lambda^2(\lambda - 3) = (-1)^3(\lambda - 0)^2(\lambda - 3)$ so

char. poly. of A is $|\lambda I_3 - A| = \lambda^2(\lambda-3)^1$

$\lambda_1 = 0$, $k_1 = 2$ and $\lambda_2 = 3$, $k_2 = 1$.

Note: $g_1 = 2$ and $g_2 = 1$ were found.

Ex: For $A = \begin{bmatrix} a_{11} & \cdots & & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ O & & & a_{nn} \end{bmatrix}$ upper triangular, char poly is

$$|\lambda I_n - A| = \begin{vmatrix} \lambda - a_{11} & \cdots & & -a_{1n} \\ & \lambda - a_{22} & & \vdots \\ & & \ddots & \vdots \\ O & & & \lambda - a_{nn} \end{vmatrix} = \prod_{i=1}^{n} (\lambda - a_{ii})$$

So, for example;

$$\begin{vmatrix} \lambda-2 & & & & \\ & \lambda-2 & & * & \\ & & \lambda+5 & & \\ & 0 & & \lambda+5 & \\ & & & & \lambda-6 \end{vmatrix} = (\lambda-2)^2(\lambda+5)^2(\lambda-6)$$

has degree $n=5$ but only 3 distinct roots; $\lambda_1=2$, $\lambda_2=-5$, $\lambda_3=6$ with alg. mults. $k_1=2$, $k_2=2$, $k_3=1$.

To find e-values of A, must factor char. poly. to get its roots. Some polys. don't factor over $\mathbb{R}$ into all linear factors.

Th: If A is similar to B, so $B = P^{-1}AP$ for some invertible P, then $|B| = |A|$.

Pf: $|P^{-1}AP| = |P^{-1}| \cdot |A| \cdot |P| = \frac{1}{|P|} |A| \cdot |P| = |A|$

since this is a product of real numbers. □

Th: If A is similar to B, then they have the same char. poly., $|\lambda I_n - A| = |\lambda I_n - B|$.

Pf: $|\lambda I_n - B| = |\lambda I_n - P^{-1}AP| = |P^{-1}\lambda I_n P - P^{-1}AP|$

$= |P^{-1}(\lambda I_n - A)P| = |P^{-1}| \cdot |\lambda I_n - A| \cdot |P| = |\lambda I_n - A|,$

since this is a product of two real numbers with a polynomial in λ. □

Note: Suppose A is diagonalizable and

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_2 & \\ & & & & & & \ddots \\ & & & & & & & \lambda_r \\ & & & & & & & & \ddots \\ & & & & & & & & & \lambda_r \end{bmatrix}$$

for distinct e-values $\lambda_1, \dots, \lambda_r$ with λ_i repeated g_i times, $1 \le i \le r$. Then

$$|\lambda I_n - A| = |\lambda I_n - D| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{g_i}$$

so $g_i = k_i$ for all $1 \le i \le r$,

geom. mult. = alg. mult. for all e-values.