Th. Suppose $|\lambda I_n - A| = \prod_{i=1}^{r} (\lambda - \lambda_i)^{k_i}$ for distinct $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, and recall that $g_i = \dim(A_{\lambda_i})$ = geom. mult. of λ_i for A. Then $1 \leq g_i \leq k_i$ for each $1 \leq i \leq r$.

__Cor__: A is diag-able iff each $g_i = k_i$.

__Practical Application__: If $k_i = 1$ then $g_i = 1$, but if $1 < k_i$ there is a chance that $1 \leq g_i < k_i$. Check __largest__ k_i first. If any $g_i < k_i$ then A __not__ diag-able, can stop process. Don't waste time on finding other e-vectors if A not diag-able.

Def: For $L: V \longrightarrow V$, S any basis of V, $A = {}_S[L]_S$, let char. poly. of L be $|\lambda I_n - A|$.

If T is any other basis of V, let $B = {}_T[L]_T$ so $B = P^{-1}AP$ for $P = {}_S P_T$ (transition matrix). Then we know $|\lambda I_n - B| = |\lambda I_n - A|$ is the same char. poly. giving a consistent definition of char. poly. for L.

<u>Possible Notations</u>: $Char_A(\lambda) = |\lambda I_n - A|$

$= Char_L(\lambda)$

Some books use $\Delta_A(\lambda)$ or $p_A(\lambda)$ for char. poly of A.

When does a quadratic poly factor into 2 linear factors?

Let poly. be $a\lambda^2+b\lambda+c$. Quadratic formula for roots is $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ which is <u>real</u> iff $b^2-4ac \geq 0$.

<u>discriminant</u> of poly is b^2-4ac.

If $b^2-4ac<0$ then get only a pair of <u>complex</u> roots, not real roots, so poly does not factor over $\mathbb{R}$.

If $b^2-4ac=0$ get one real root, repeated like $\lambda^2-4\lambda+4=(\lambda-2)^2$

$b^2-4ac=16-4(1)(4)=0$

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ so $A - \lambda I_3 = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix}$

$$|A - \lambda I_3| = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = -(\lambda-1)\left((1-\lambda)^2+1\right)$$

$$= -(\lambda-1)(\lambda^2-2\lambda+2)$$

(cofactor expansion along row 2)

but $b^2-4ac = 4-4(1)(2) = -4 < 0$

so $\lambda^2-2\lambda+2$ has no real roots.

only got one real e-value, $\lambda_1 = 1$, $K_1 = 1$, $g_1 = 1$

could not get an e-basis of $\mathbb{R}^3$ for A.

This A is not diag-able over $\mathbb{R}$.

<u>Ex</u>: $A = \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$ so $A-\lambda I_3 = \begin{bmatrix}1-\lambda&0&1\\0&1-\lambda&0\\1&0&1-\lambda\end{bmatrix}$

$|A-\lambda I_3| = (1-\lambda)\begin{vmatrix}1-\lambda&1\\1&1-\lambda\end{vmatrix} = -(\lambda-1)\left((1-\lambda)^2-1\right)$

$= -(\lambda-1)(\lambda^2-2\lambda) = -(\lambda-1)(\lambda)(\lambda-2)$ has three distinct roots: order by inc. size:

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, alg. mult.s: $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$, so geom. mults: $g_1 = 1$, $g_2 = 1$, $g_3 = 1$. Guaranteed to get one e-basis vector for each e-value, indep, e-basis of $\mathbb{R}^3$ for A. Find P s.t.

$P^{-1}AP = D = \begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}$ is diagonal.

$\boxed{\lambda_1 = 0}$: Get A_0: Solve $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -r$
$x_2 = 0$
$x_3 = r \in \mathbb{R}$

$A_0 = \left\{ \begin{bmatrix} -r \\ 0 \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ has basis $T_1 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\boxed{\lambda_2 = 1}$ Get A_1: Solve $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = r \in \mathbb{R}$, $x_3 = 0$

$A_1 = \left\{ \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ basis $T_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\boxed{\lambda_3 = 2}$: Get A_2: Solve $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = r$, $x_2 = 0$, $x_3 = r \in \mathbb{R}$

$A_2 = \left\{ \begin{bmatrix} r \\ 0 \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\}$ has basis $T_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

So e-basis $T = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and

$P = {}_S P_T = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ should make $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$

Easier to check that $\boxed{AP = PD}$

$$\underset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{P}{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \overset{\checkmark}{=} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \overset{\checkmark}{=} \underset{P}{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}.$$

<u>Pf.</u> Show that $g_i \leq k_i$ where $\mathrm{Char}_A(\lambda) = \prod_{i=1}^{r}(\lambda-\lambda_i)^{k_i}$ and $g_i = \dim(A_{\lambda_i})$. We can just do the case $g_1 \leq k_1$ since the rest are similar. Let $T_1 = \{w_{11}, \dots, w_{1g_1}\}$ be a basis of A_{λ_1} and extend T_1 to a basis of F^n, $T = \{w_{11}, \dots, w_{1g_1}, v_1, \dots, v_{n-g_1}\}$. Since $Aw_{1j} = \lambda_1 w_{1j}$ for $1 \leq j \leq g_1$, $L_A : F^n \to F^n$ has values $L_A(w_{1j}) = Aw_{1j} = \lambda_1 w_{1j}$ so $[L_A(w_{1j})]_T = \lambda_1 e_j$ means the matrix $B = {}_T[L_A]_T = P^{-1}AP$ for $P = {}_S P_T$ has a block form $B = \left[\begin{array}{c|c} \lambda_1 I_{g_1} & C \\ \hline O & D \end{array}\right]$.

Then $\det(\lambda I_n - B) = \det \left[\begin{array}{c|c} (\lambda - \lambda_1) I_{g_1} & -C \\ \hline 0 & (\lambda I_{n-g_1} - D) \end{array} \right]$

$= \det((\lambda - \lambda_1) I_{g_1}) \cdot \det(\lambda I_{n-g_1} - D)$

$= (\lambda - \lambda_1)^{g_1} \det(\lambda I_{n-g_1} - D)$ because of the block (upper) triangular form. $\det(\lambda I_{n-g_1} - D) = \mathrm{Char}_D(\lambda)$ is some poly. of degree $n - g_1$ which may or may not contain more factors of $(\lambda - \lambda_1)$, but we certainly have at least $(\lambda - \lambda_1)^{g_1}$ as a factor in $\det(\lambda I_n - B) = \det(\lambda I_n - A) = \mathrm{Char}_A(\lambda)$, so $g_1 \leq k_1$.

Sums and Direct sums of subspaces:

Def. For $W_1, \dots, W_t \leq V$ (subspaces of V), let
$W_1 + \dots + W_t = \{w_1 + \dots + w_t \in V \mid w_i \in W_i \text{ for } 1 \leq i \leq t\}$ be
their sum. Say the sum is <u>direct</u> when each
vector in the sum has a unique expression as
above (keeping the order of terms fixed).

Th. $\sum_{i=1}^{t} W_i \leq V$ and the sum is direct iff
$W_i \cap \left(\sum_{j \neq i} W_j\right) = \{\theta\}$ for each $1 \leq i \leq t$.

Pf. It is easy to check closure of the sum under
$+$ and $\cdot$, and that θ is in it. Let $w = \sum_{i=1}^{t} w_i = \sum_{i=1}^{t} w_i'$
then for each $1 \leq i \leq t$ we have $w_i = w_i'$ if sum is dir.

If $w_i = \sum_{j \neq i} w_j \neq \theta$ it would contradict the uniqueness of expression. Conversely, assuming all those intersections are $\{\theta\}$ prove the sum is direct as follows. Let $\sum_{i=1}^{t} w_i = \sum_{i=1}^{t} w_i'$ be two distinct expressions for the same w. Say $w_1 \neq w_1'$ so $w_1 - w_1' = \sum_{i=2}^{t} (w_i' - w_i) \neq \theta$ but the leftside is in W_1 and the right side is in $\sum_{i=2}^{t} W_i$ so the intersection is non-trivial. $\square$

<u>Note</u>: For $t=2$, $W_1 + W_2$ is a direct sum iff $W_1 \cap W_2 = \{\theta\}$ but for $t=3$, $W_1 + W_2 + W_3$ is direct iff $W_1 \cap (W_2 + W_3) = \{\theta\} = W_2 \cap (W_1 + W_3) = W_3 \cap (W_1 + W_2)$

Notation: When a sum of subspaces is direct we write it $W_1 \oplus W_2 \oplus \cdots \oplus W_t = \bigoplus_{i=1}^{t} W_i$

Th: Suppose $T_i = \{w_{i1}, \dots, w_{id_i}\}$ is a basis of W_i. Then $W = \sum_{i=1}^{t} W_i$ is direct iff $T_1 \cup \cdots \cup T_t$ (as a list) is a basis for W, so $\dim(W) = \sum_{i=1}^{t} d_i$.

Pf. Exercise, probably in our textbook.

Cor. If $L : V \to V$ has distinct e-values $\lambda_1, \dots, \lambda_r$ and $L_{\lambda_1}, \dots, L_{\lambda_r}$ are the corresponding e-spaces, then their sum is direct, $L_{\lambda_1} \oplus \cdots \oplus L_{\lambda_r} \leq V$.

Def. For $L : V \to V$ say subsp. $W \leq V$ is <u>L-invariant</u> when $L(W) \subseteq W$. Then $L|_W : W \to W$ is def'd $L|_W(w) = L(w)$ $\forall w \in W$

<u>Th</u>: Suppose $L: V \to V$ and $V = W_1 \oplus \cdots \oplus W_t$ for L-invar. subspaces $W_i \leq V$. Let $L|_{W_i} = L_i$, and let T_i be a basis is for W_i, $T = T_1 \cup \cdots \cup T_t$.

Then ${}_T[L]_T = \left[\begin{array}{c|c|c|c} {}_{T_1}[L_1]_{T_1} & 0 & & 0 \\ \hline 0 & {}_{T_2}[L_2]_{T_2} & & 0 \\ \hline 0 & 0 & \ddots & \\ \hline & & & {}_{T_t}[L_t]_{T_t} \end{array}\right]$ is the block diagonal matrix whose blocks are ${}_{T_i}[L_i]_{T_i}$ for $1 \leq i \leq t$.

<u>Pf</u>. Follows from the algorithm to find ${}_T[L]_T$ and for each ${}_{T_i}[L_i]_{T_i}$.

Th: Let $W_1, W_2 \leq V$ be fin.dim'l subspaces.

Then $W_1 + W_2$ is fin. dim'l and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Pf. Let $W_1 \cap W_2$ have basis $R = \{u_1, \dots, u_r\}$.

Extend R to a basis of W_1: $S = \{u_1, \dots, u_r, v_1, \dots, v_s\}$.

Extend R to a basis of W_2: $T = \{u_1, \dots, u_r, w_1, \dots, w_t\}$

Claim: $U = \{u_1, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_t\}$ is a basis of $W_1 + W_2$. If so, this gives the result we want:

$$\dim(W_1 + W_2) = r + s + t = (r+s) + (r+t) - r$$
$$= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

U spans $W_1 + W_2$ since any element in $W_1 + W_2$ is a sum $\left(\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j\right) + \left(\sum_{i=1}^{r} c_i u_i + \sum_{k=1}^{t} d_k w_k\right) \in \langle U \rangle$.

To show $\mathcal{V}$ is indep, suppose

$$\theta = \sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j + \sum_{k=1}^{t} c_k w_k$$ then $v =$

$$\underbrace{-\sum c_k w_k}_{\in W_2} = \underbrace{\sum a_i u_i + \sum b_j v_j}_{\in W_1} \in W_1 \cap W_2$$ since so $v \in \langle R \rangle$ can be written as $v = \sum_{i=1}^{r} d_i u_i$ giving $\theta = \sum d_i u_i + \sum c_k w_k$ a dep. rel. on basis T so all $d_i = 0$ and all $c_k = 0$. Then $\theta = \sum a_i u_i + \sum b_j v_j$ is a dep. rel. on basis S, so all $a_i = 0$ and all $b_j = 0$. This gives indep. of $\mathcal{V}$. □

__Application.__ If $\dim(V) = 10$, $\dim(W_1) = 7$ and $\dim(W_2) = 6$, what are all possibilities for $\dim(W_1 \cap W_2)$?