1215

<u>Solution</u>: $\dim(W_1+W_2) \leq \dim(V) = 10$ so

$\dim(W_1)+\dim(W_2)-\dim(W_1\cap W_2) \leq \dim(V) = 10$ so

$7 + 6 - \dim(W_1\cap W_2) \leq 10$ so

$3 = 13-10 = 7+6-10 \leq \dim(W_1\cap W_2)$.

Also, $W_1\cap W_2 \leq W_1$ and $W_1\cap W_2 \leq W_2$ so

$\dim(W_1\cap W_2) \leq \mathrm{Min}(\dim(W_1), \dim(W_2))$

$= \mathrm{Min}(7,6) = 6$.

Possibilities are: $3 \leq \dim(W_1\cap W_2) \leq 6$

Unfinished business about det:

Recall: For $A=[a_{ij}]\in F_n^n$ we defined cofactors $A_{rs}=(-1)^{r+s}\det(M_{rs})$ for each $1\le r,s\le n$. We can get a new matrix $B=[A_{ij}]\in F_n^n$ from these numbers, but we really want its transpose.

Def. The classical adjoint of $A\in F_n^n$ is $\operatorname{adj}(A)=B^T=[A_{ji}]$.

Ex: $n=2$: $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ so $B=[A_{ij}]=\begin{bmatrix}a_{22} & -a_{21}\\ -a_{12} & a_{11}\end{bmatrix}$

so $\operatorname{adj}(A)=B^T=\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$. Note

$$A\operatorname{adj}(A)=\begin{bmatrix}(a_{11}a_{22}-a_{12}a_{21}) & 0\\ 0 & (a_{11}a_{22}-a_{12}a_{21})\end{bmatrix}=(\det A)I_2$$

To see why $A\,\mathrm{adj}(A) = (\det A)\, I_n$ is true for any $n \geq 2$, go back to cofactor expansions and prove this result:

<u>Th.</u> For $A = [a_{ij}] \in F_n^n$, with cofactors A_{rs}, we have

(a) $\delta_{rs} \det(A) = \sum_{j=1}^{n} a_{rj} A_{sj}$ and

(b) $\delta_{rs} \det(A) = \sum_{i=1}^{n} a_{ir} A_{is}$.

<u>Pf.</u> When $r = s$ this is the usual cofactor expansion theorem for computing $\det(A)$. When $r \neq s$ the right hand sides are cofactor expansions for the matrix obtained from A by replacing row (col.) s by row (col. r), so those det's are 0 (two ident. rows or cols).

<u>Cor.</u> $\forall A \in F_n^n$, $A\,\mathrm{adj}(A) = \det(A)\, I_n = \mathrm{adj}(A)\, A$.

<u>Pf.</u> The (r,s)-entry of $A\,\mathrm{adj}(A)$ is $\mathrm{Row}_r(A)\,\mathrm{Col}_s(\mathrm{adj}(A)) = \sum_{j=1}^{n} a_{rj} A_{sj} = \delta_{rs} \det(A)$. The second equality, $\mathrm{adj}(A)\, A = (\det A)\, I_n$ comes from (b) of the Theorem above. $\square$

<u>Th</u>: (Cayley-Hamilton). For any $A \in F_n^n$, the char. poly. of A, $\mathrm{Char}_A(t) = \Delta(t) = \det(tI_n - A)$, is satisfied by A, that is, $\Delta(A) = 0_n^n$.

<u>Pf</u>: (FAKE!) $\Delta(A) = \det(A I_n - A) = \det(0_n^n) = 0$.

WHY IS THIS NONSENSE?

If we write out the char. poly. of $A=[a_{ij}]$,

$$\Delta_A(t)=\Delta(t)=\det(tI_n-A)=t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0$$

$$=\det\begin{bmatrix} t-a_{11} & \cdots & -a_{ij} \\ \vdots & t-a_{22} & \vdots \\ -a_{ji} & \cdots & t-a_{nn}\end{bmatrix}=\det\left[t\delta_{ij}-a_{ij}\right]$$

it clearly makes no sense to try to "plug in" $A\in F_n^n$ for t in the $n\times n$ matrix tI_n-A.

Let $B(t)=\operatorname{adj}(tI_n-A)$, classical adjoint, whose entries are cofactors of tI_n-A, so are polys. in t of degree at most $n-1$. We can write $B(t)=B_{n-1}t^{n-1}+\cdots+B_1t+B_0$ for $B_i\in F_n^n$ constant matrices not involving t. Since $(tI_n-A)B(t)=\det(tI_n-A)I_n$ we have

$$(tI_n-A)(B_{n-1}t^{n-1}+\cdots+B_1t+B_0)=(t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0)I_n$$

Multiply out the left side, group terms by power of t and compare with corresponding terms on the right side:

$$I_n B_{n-1} t^n + (I_n B_{n-2} - AB_{n-1}) t^{n-1} + (I_n B_{n-3} - AB_{n-2}) t^{n-2} +$$
$$\cdots + (I_n B_1 - AB_2) t^2 + (I_n B_0 - AB_1) t - AB_0 \quad =$$
$$I_n t^n + \quad a_{n-1} I_n t^{n-1} + a_{n-2} I_n t^{n-2} +$$
$$\cdots + \quad a_2 I_n t^2 + \quad a_1 I_n t \quad + a_0 I_n \text{, gives:}$$

$$\left.\begin{aligned} B_{n-1} &= I_n \\ B_{n-2} - AB_{n-1} &= a_{n-1} I_n \\ B_{n-3} - AB_{n-2} &= a_{n-2} I_n \\ &\vdots \\ B_1 - AB_2 &= a_2 I_n \\ B_0 - AB_1 &= a_1 I_n \\ -AB_0 &= a_0 I_n \end{aligned}\right\} \text{so} \left\{\begin{aligned} A^n B_{n-1} &= A^n \\ A^{n-1} B_{n-2} - A^n B_{n-1} &= a_{n-1} A^{n-1} \\ A^{n-2} B_{n-3} - A^{n-1} B_{n-2} &= a_{n-2} A^{n-2} \\ &\vdots \\ A^2 B_1 - A^3 B_2 &= a_2 A^2 \\ AB_0 - A^2 B_1 &= a_1 A \\ -AB_0 &= a_0 I_n \end{aligned}\right.$$

The sum of all terms on the left sides of these equations is 0^n_n and the sum on the right sides is $A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I = \Delta(A)$, so $\Delta(A) = 0^n_n$ gives the result. $\square$ [22]

<u>Cor.</u> Let $L: V \to V$, $\dim(V) = n$, $A = {}_S[L]_S \in F^n_n$ for some basis S of V, $\mathrm{Char}_L(t) = \mathrm{Char}_A(t) = \det(tI_n - A)$. then $\mathrm{Char}_L(L) = 0^V_V \in \mathrm{End}(V)$.

<u>Discussion</u>: In $\mathcal{V} = \mathrm{End}(V) = \mathrm{Lin}(V,V)$, the set $\{I_V, L, L^2, \dots, L^{n^2}\}$ is dep. since $\dim(\mathcal{V}) = n^2$ if $\dim(V) = n$. So there is some smallest $1 \le m \in \mathbb{Z}$ s.t. $\{I_V, L, \dots, L^m\}$ is dep. but $\{I_V, L, \dots, L^{m-1}\}$ is indep.

Similarly; In F^n_n the set $\{I_n, A, \dots, A^{n^2}\}$ is dep.

since dim(F_n^n) $= n^2$. Cayley-Hamilton Thm says $\{I_n, A, \dots, A^n\}$ is dep. so that smallest m s.t. $\{I_n, A, \dots, A^m\}$ is dep. but $\{I_n, A, \dots, A^{m-1}\}$ is indep. must be $m \le n$.

__Def.__ For $A \in F_n^n$, a __minimal poly.__ for A is a poly. $m_A(t) \in F[t]$ which is __monic__ (top coeff. 1) and is satisfied by A, and is the smallest possible degree. For $L: V \to V$ the corresponding minimal poly for L is denoted $m_L(t)$.

__Recall discussion on page 70.__

We have now answered some of those questions.

For similar $B = P^{-1}AP$ what is the relation between

$m_A(t)$ and $m_B(t)$?

Is $m_A(t)$ uniquely determined by A ?

<u>Lemma</u>: Let $B = P^{-1}AP$ and $f(t) \in F[t]$. Then $f(B) = P^{-1}f(A)P$.

<u>pf.</u> For each $0 \le i \in \mathbb{Z}$, $B^i = P^{-1}A^iP$ can be proven by induction on i. For $i=0$, $B^0 = I_n = P^{-1}I_nP = P^{-1}A^0P$.

Assuming the equation is true for some $0 \le i \in \mathbb{Z}$,

$B^{i+1} = (P^{-1}AP)^{i+1} = (P^{-1}AP)^i(P^{-1}AP) = (P^{-1}A^iP)(P^{-1}AP)$
$= P^{-1}A^i(PP^{-1})AP = P^{-1}A^iI_nAP = P^{-1}A^iAP = P^{-1}A^{i+1}P$.

Say $f(t) = \sum_{i=0}^{m} c_i t^i$ so $f(B) = \sum_{i=0}^{m} c_i B^i = \sum_{i=0}^{m} c_i (P^{-1}A^iP)$
$= P^{-1}\left(\sum_{i=0}^{m} c_i A^i\right)P = P^{-1}f(A)P$. $\square$

So if $f(A)=0_n^n$ then $f(B)=P^{-1}f(A)P=0_n^n$, says similar matrices are satisfied by the same polys.

<u>Th</u>: Let $A\in F_n^n$ and let $\mathcal{I}_A=\{f(t)\in F[t] \mid f(A)=0_n^n\}$. Then $\mathcal{I}_A \trianglelefteq F[t]$ is an ideal in the ring $F[t]$ and $\mathcal{I}_A=\{g(t)m_A(t)\in F[t] \mid g(t)\in F[t]\}=(m_A(t))$ is the (principal) ideal of all multiples of $m_A(t)$, the unique monic poly of minimal degree in $\mathcal{I}_A$.

<u>Pf</u>. It is clear that $\mathcal{I}_A$ is closed under + and $\cdot$, so it is a subspace of $F[t]$, and $\mathcal{I}_A$ absorbs products, $\forall f(t)\in\mathcal{I}_A$, $\forall g(t)\in F[t]$, $f(t)\cdot g(t)\in\mathcal{I}_A$, so it is an ideal. The Euclidean Algorithm in $F[t]$ (see page 66)

implies that any ideal of $F[t]$ is principal, 225
but here we apply it only to $\mathcal{I}_A$. Let $m_A(t)$ be a poly. of minimal degree in $\mathcal{I}_A$. Multiplying by a scalar we can assume it is monic. Then $\forall f(t) \in \mathcal{I}_A$, $\exists q(t), r(t) \in F[t]$ s.t. $f(t) = q(t) m_A(t) + r(t)$ where either $r(t) = 0$ or $\deg(r(t)) < \deg(m_A(t))$. Evaluation of both sides at $t = A$ gives $0^n_n = f(A) = q(A)\, m_A(A) + r(A)$ and $m_A(A) = 0^n_n$ so $0^n_n = r(A)$. If $\deg(r(t)) < \deg(m_A(t))$ we get a contradiction to $m_A(t)$ being the smallest degree poly. in $\mathcal{I}_A$ so $r(t)$ must be the zero poly. The uniqueness of $m_A(t)$ follows because any two polys of min. degree in $\mathcal{I}_A$ must divide each other. They can only differ by a scalar multiple. Monic $\Rightarrow$ Unique. ∎