Th: Suppose for $A \in F^n_n$ that $\mathrm{Char}_A(t) = \prod_{i=1}^{r} (t-\lambda_i)^{k_i}$ for $\lambda_1, \dots, \lambda_r \in F$ the distinct e-values of $A$. Then $m_A(t) = \prod_{i=1}^{r} (t-\lambda_i)^{m_i}$ for some $1 \leq m_i \leq k_i$.

Pf. By Cayley-Hamilton Th. $\mathrm{Char}_A(t) \in F[t]$ is satisfied by $A$ so it is in $\mathcal{I}_A$ so it is a multiple of $m_A(t)$. Then for each $1 \leq i \leq r$, $(t-\lambda_i)^{m_i}$ divides $\mathrm{Char}_A(t)$ so $m_i \leq k_i$. Suppose $m_1 = 0$. Since $k_1 \geq 1$ $\lambda_1$ is an e-value of $A$ so $A_{\lambda_1} \neq \{0^n_1\}$. Let $0^n_1 \neq X \in A_{\lambda_1}$ so $(A-\lambda_1 I_n)X = 0^n_1$ but for all $1 < i \leq r$, $(A-\lambda_i I_n)X \neq 0^n_1$

In fact, $(A-\lambda_i I_n)X = AX - \lambda_i X = \lambda_1 X - \lambda_i X = (\lambda_1 - \lambda_i)X$

so $\prod_{i=2}^{r} (A-\lambda_i I_n)^{m_i} X = \prod_{i=2}^{r} (\lambda_1 - \lambda_i)^{m_i} X \neq 0^n_1$ contradicts $m_A(A) = 0^n_n$ ∎

<u>Def</u>. Let $L : V \to V$, $\dim(V) = n$, $A = {}_S[L]_S$ for some basis $S$ of $V$, $\mathrm{Char}_A(t) = \prod_{i=1}^{r} (t-\lambda_i)^{k_i}$ and $m_A(t) = \prod_{i=1}^{r} (t-\lambda_i)^{m_i}$ the char. and min. polys, resp. Let $W_i = \mathrm{Ker}\left((L - \lambda_i I_V)^{m_i}\right) \leq V$ for $1 \leq i \leq r$, be the "<u>generalized eigenspace</u>" of $L$ for $\lambda_i$.

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<u>Th</u>: $V = W_1 \oplus \cdots \oplus W_r$.

<u>Pf</u>. If $r = 1$ so $m_A(t) = (t-\lambda_1)^{m_1}$ and $L$ has only one e.value, then $0_V^V = m_A(L) = (L - \lambda_1 I_V)^{m_1}$ means $V = W_1$. Suppose $r \geq 2$. Let $f(t) = (t-\lambda_1)^{m_1}$ and $g(t) = \prod_{i=2}^{r} (t-\lambda_i)^{m_i}$ so $\gcd(f(t), g(t)) = 1$ and then $\exists\, a(t), b(t) \in F[t]$ s.t. $a(t)f(t) + b(t)g(t) = 1$.

Evaluation at $t = L$ gives $a(L)f(L) + b(L)g(L) = I_V$
so $\forall v \in V$, $v = a(L)f(L)v + b(L)g(L)v$.
Notice that $f(L)g(L) = m_A(L) = 0_V^V$ is the zero map
so $b(L)g(L)v \in \mathrm{Ker}(f(L)) = W_1$ and
$a(L)f(L)v \in \mathrm{Ker}(g(L))$, which means
$V = W_1 + \mathrm{Ker}(g(L))$. If $v \in W_1 \cap \mathrm{Ker}(g(L))$
then $v = a(L)f(L)v + b(L)g(L)v = \theta_V + \theta_V = \theta_V$
So $V = W_1 \oplus \mathrm{Ker}(g(L))$ is a direct sum of $L$-invariant subspaces of $V$. The restriction of $L$ to $W_1$ has min. poly. $f(t)$ and the restriction of $L$ to $\mathrm{Ker}(g(L))$ has min. poly. $g(t)$. By induction on $r$, $\mathrm{Ker}(g(L)) = W_2 \oplus \cdots \oplus W_r$. $\square$

<u>Cor.</u> $L: V \to V$ with $m_L(t) = \prod_{i=1}^{r} (t-\lambda_i)^{m_i}$ is diag-able iff $m_i = 1$ for $1 \le i \le r$.

<u>Pf.</u> As we saw on page 199, if $L$ is diag-able it means ${}_T[L]_T = D = P^{-1}AP$ for one basis $T$ of $V$ so $\mathrm{Char}_L(t) = \mathrm{Char}_A(t) = \mathrm{Char}_D(t) = \prod_{i=1}^{r} (t-\lambda_i)^{g_i}$ for $g_i = \dim(L_{\lambda_i}) = k_i$. From the block form of

$$D = \begin{bmatrix} \lambda_1 I_{g_1} & & \\ & \ddots & \\ & & \lambda_r I_{g_r} \end{bmatrix}$$

we get $D - \lambda_i I_n$ has $O_{g_i}^{g_i}$ in its $i^{\text{th}}$ block, so the product $(D-\lambda_1 I_n)(D-\lambda_2 I_n)\cdots(D-\lambda_r I_n) = O_n^n$ giving all $m_i = 1$.

Conversely, from the Theorem above, if all $m_i = 1$ then $W_i = \mathrm{Ker}(L - \lambda_i I_V) = L_{\lambda_i}$ so $V = L_{\lambda_1} \oplus \cdots \oplus L_{\lambda_r}$ shows $L$ diag-able. ∎

**Quotient Spaces:** As in other parts of algebra, quotient structures are useful and important in linear algebra. They can be used for inductive proofs and other applications. 230

**Def.** For $W \leq V$ define a relation on $V$,
$v_1 \equiv v_2 \pmod{W}$ when $v_1 - v_2 \in W$.

**Th.** $\equiv \pmod{W}$ is an equivalence relation.

**Def.** $V/W = \{[v]_W \mid v \in V\}$ where the equiv. class $[v]_W = \{x \in V \mid x \equiv v \pmod{W}\}$. Define $+$ and $\cdot$ on $V/W$ by $[v]_W + [v']_W = [v+v']_W$ and $\forall \alpha \in F$ $\alpha \cdot [v]_W = [\alpha \cdot v]_W$. Let $\theta_{V/W} = [\theta_V]_W$. (Check $+, \cdot$ well-defined)

**Th:** $(V/W, +, \cdot, \theta_{V/W})$ is a vector space over $F$.

Let $\pi_W : V \to V/W$ be the natural surjection $\pi_W(v) = [v]_W$ which is linear by def. of $+$ and $\cdot$.

**Th**: Let $L : V \longrightarrow W$, $K = \mathrm{Ker}(L)$, $U \leq V$. There is a lin. map $\bar{L} : V/U \to W$ s.t. $L = \bar{L} \circ \pi_U$ iff $U \leq K$.

**Pf**. The diagram for this situation is

$$
\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "\pi_U"'] & W \\
V/U \arrow[ur, "\bar{L}"'] &
\end{tikzcd}
$$

The requirement is that $\forall v \in V$, $L(v) = \bar{L}([v]_U)$ is a consistent definition of $\bar{L}$. Suppose $[v]_U = [v']_U$ for $v \in V$ so to be "well-defined" we need $L(v) = L(v')$. So $\bar{L}$ is well-defined when $v - v' \in U \Rightarrow L(v - v') = \theta_W$, that is, when $U \leq \mathrm{Ker}(L)$. The lin. of $\bar{L}$ is clear. $\square$

Th (1st Isom. Th.) If $L: V \to W$ is onto and $K = Ker(L)$, then $\bar{L}: V/K \to W$ is an isomorphism.

Pf. First check $\bar{L}$ is injective whether or not $L$ is onto. If $\bar{L}([v]_K) = \theta_W$ then $L(v) = \theta_W$ so $v \in K$ so $[v]_K = [\theta]_K = \theta_{V/K}$ so $Ker(\bar{L}) = \{[\theta]_K\}$ is trivial and $\bar{L}$ is injective.

If $L$ is onto, so is $\bar{L}$ since $Range(\bar{L}) = Range(L)$.

Thus, $\bar{L}$ is bijective, an isomorphism.

Def. Say vectorspaces $V$ and $W$ are isomorphic when $\exists\ L: V \to W$ an isomorphism. Notation: $V \cong W$.

If $L: V \to W$ is any linear map, $V/Ker(L) \cong Range(L)$

There are other isomorphism theorems.