Th. (2nd Isom. Th.) Let $W \leq V$, $\pi_W : V \to V/W$. Each subspace of V/W is of the form $U/W = \{[u]_W \in V/W \mid u \in U\}$ for $W \leq U \leq V$. There is an isomorphism $(V/W)/(U/W) \cong V/U$.

Th: Suppose $L : V \to V$ and $W \leq V$ is L-invariant. Then there is an induced lin. map $\bar{L} : V/W \to V/W$ such that $\bar{L}([v]_W) = [L(v)]_W$.

Pf. The diagram for this situation is:

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi_W"] & V \arrow[d, "\pi_W"] \\ V/W \arrow[r, "\bar{L}"] & V/W \end{tikzcd}$$

To check if the formula is a well-defined condition suppose $[v]_W = [v']_W$ so $v - v' \in W$ so $L(v) - L(v') \in W$ so $[L(v)]_W = [L(v')]_W$. $\square$

Th. Let $L : V \to V$, $W \leq V$ an L-invar. subsp.,
$R = \{v_1, \dots, v_r\}$ a basis of W, extended to a basis
$S = \{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$ of V. Then $T = \{[v_{r+1}]_W, \dots, [v_n]_W\}$
is a basis of V/W and ${}_S[L]_S = \left[\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right] \in F_n^n$ where
$A = {}_R[L|_W]_R \in F_r^r$ and $D = {}_T[\bar{L}]_T$.

Pf. Exercise.

Cor. If $f(t) \in F[t]$ and $f(L) = O_V^V$ then $f(\bar{L}) = O_{V/W}^{V/W}$
with notations and assumptions as in the Theorem above.

Pf. $\forall v \in V$, $f(L)v = \theta_V$ so $f(\bar{L})[v]_W = [f(L)v]_W = [\theta_V]_W$
From the matrix point of view, $f({}_S[L]_S)$ has block form

$$\left[\begin{array}{c|c} f(A) & * \\ \hline 0 & f(D) \end{array}\right] = \left[\begin{array}{c|c} O_r^r & O \\ \hline O & O_{n-r}^{n-r} \end{array}\right]$$

so $f(A) = O_r^r$ and $f(D) = O_{n-r}^{n-r}$.

$\square$

<u>Cor</u>. With notations and assumptions as in the Theorem above, $\mathrm{Char}_L(t) = \mathrm{Char}_{L|W}(t)\cdot \mathrm{Char}_{\bar{L}}(t) = \mathrm{Char}_A(t)\cdot \mathrm{Char}_D(t)$. Also, $m_{\bar{L}}(t) = m_D(t)$ divides $m_L(t)$.

<u>Pf</u>. $\mathrm{Char}_L(t) = \det(tI_n - {}_S[L]_S) = \det\left[\begin{array}{c|c} tI_r - A & -B \\ \hline 0 & tI_{n-r} - D\end{array}\right]$
$= \det(tI_r - A)\cdot \det(tI_{n-r} - D) = \mathrm{Char}_A(t)\cdot \mathrm{Char}_D(t)$.
From last Cor., since $m_L(L) = 0_V^V$ get $m_L(\bar{L}) = 0_{V/W}^{V/W}$ so $m_{\bar{L}}(t)$ divides $m_L(t)$. □

<u>Th</u>: Suppose $A = \mathrm{diag}(B_1, \dots, B_r) \in F_n^n$ is in block diagonal form. Then $\mathrm{Char}_A(t) = \prod_{i=1}^{r} \mathrm{Char}_{B_i}(t)$ and $m_A(t) = \mathrm{l.c.m.}(m_{B_1}(t), \dots, m_{B_r}(t))$.

<u>Pf</u>. The first part follows as in the last proof.

The second part follows from the fact that for any $f(t) \in F[t]$, $f(A) = \mathrm{diag}(f(B_1), \dots, f(B_r))$. $\square$

Jordan Form:

Def. For $\lambda \in F$, define a Jordan block $n \times n$ matrix

$$J(\lambda, n) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} = \lambda I_n + \sum_{i=1}^{n-1} E_{i(i+1)} \in F_n^n \quad \text{for } n \geq 2$$

and $J(\lambda, 1) = [\lambda] \in F_1^1$.

Facts: $\mathrm{Char}_{J(\lambda,n)}(t) = (t-\lambda)^n = m_{J(\lambda,n)}(t)$.

The first "=" is clear. Writing $J = J(\lambda, n)$, $N = J - \lambda I_n$ is nilpotent and the least power of N that gives 0_n^n is $N^n = 0_n^n$.