Def. Say $A \in F_n^n$ is nilpotent when $A^k = 0_n^n$ for some $1 \le k \in \mathbb{Z}$. The least such k is called the "order" or "index of nilpotence" of A. We have a similar definition for $L \in End(V)$.

Th. $L: V \to V$ is represented by a Jordan block matrix ${}_T[L]_T = J(\lambda, n)$ for basis $T = \{v_1, \dots, v_n\}$ iff $L(v_1) = \lambda v_1$, $L(v_2) = \lambda v_2 + v_1, \dots, L(v_n) = \lambda v_n + v_{n-1}$, that is, v_1 is an e-vector for L with e.value λ, and for $2 \le j \le n$, $L(v_j) = \lambda v_j + v_{j-1}$.

Pf. This follows from the fact that $[L(v_j)]_T = Col_j(J(\lambda, n))$. $\square$

Note: It means $(L - \lambda I_V) v_j = v_{j-1}$ if $v_0 = \theta$.

Another viewpoint: $v_1 \in \mathrm{Ker}(L-\lambda I_V)$

$v_2 \in \mathrm{Ker}((L-\lambda I_V)^2)$ but $v_2 \notin \mathrm{Ker}(L-\lambda I_V)$

$v_3 \in \mathrm{Ker}((L-\lambda I_V)^3)$ but $v_3 \notin \mathrm{Ker}((L-\lambda I_V)^2)$

$\vdots$

$v_n \in \mathrm{Ker}((L-\lambda I_V)^n)$ but $v_n \notin \mathrm{Ker}((L-\lambda I_V)^{n-1})$

Let $W_\lambda^{(i)} = L_\lambda^{(i)} = \mathrm{Ker}((L-\lambda I_V)^i)$ be generalized eigenspaces of L for e-value λ. Let $v_n \in L_\lambda^{(n)} - L_\lambda^{(n-1)}$. Then for $1 \le i \le n-1$, $(L-\lambda I_V)^i v_n = v_{n-i}$ defines $v_{n-1}, \dots, v_1$ s.t. $T = \{v_1, \dots, v_n\}$ is a "Jordan" basis of V so that ${}_T[L]_T = J(\lambda, n)$.

Since $m_{J(\lambda,n)}(t) = (t-\lambda)^n$, we have

$$L_\lambda = L_\lambda^{(1)} \subsetneq L_\lambda^{(2)} \subsetneq \cdots \subsetneq L_\lambda^{(n-1)} \subsetneq L_\lambda^{(n)} = L_\lambda^{(n+1)} = \cdots = V$$

a strictly increasing sequence (chain) of gen. e-spaces which stabilizes at the exponent in the min. poly.

If $\dim(L_\lambda^{(n)}) - \dim(L_\lambda^{(n-1)}) > 1$ then there are indep. choices for v_n, contradicting the assumption that ${}_T[L]_T$ is a single Jordan block.

What if more J-blocks occur?

Th: Suppose $L: V \to V$ and ${}_T[L]_T = \operatorname{diag}(J(\lambda, n_1), J(\lambda, n_2), \dots, J(\lambda, n_r))$ has a block diagonal form with J-blocks all with the same e-value λ, but sizes $n_1 \geq n_2 \geq \dots \geq n_r \geq 1$ so $n = \dim(V) = n_1 + n_2 + \dots + n_r$. Then J-basis $T = \{v_{11}, \dots, v_{1n_1}, v_{21}, \dots, v_{2n_2}, \dots, v_{r1}, \dots, v_{rn_r}\}$ with $\{v_{i1} \in L_\lambda \mid 1 \leq i \leq r\}$ an e-basis of $L_\lambda = L_\lambda^{(1)}$ so $g_\lambda = r$ is the geom. mult. of λ for L, and for $2 \leq j \leq n_1$, $\{v_{ij} \in T \mid 1 \leq i \leq r\}$ is a basis of $L_\lambda^{(j)}$. If $j > n_i$ there is no v_{ij} in T.

Example: Suppose $L: V \to V$ is rep'ed by [2 41

J-block matrix

$$J = \left[\begin{array}{ccc|cc} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{array}\right] = \left[\begin{array}{c|c} J(\lambda,3) & 0 \\ \hline 0 & J(\lambda,2) \end{array}\right] = {}_T[L]_T \text{ and}$$

$$T = \{v_{11}, v_{12}, v_{13}, v_{21}, v_{22}\}$$

Then $L(v_{11}) = \lambda v_{11}$, $L(v_{21}) = \lambda v_{21}$, $L_\lambda = \underbrace{\langle v_{11}, v_{21} \rangle}_{\text{basis}}$

$g_\lambda = 2 = \dim(L_\lambda)$. $\quad Char_J(t) = (t-\lambda)^5$

$L(v_{12}) = \lambda v_{12} + v_{11}$

$L(v_{13}) = \lambda v_{13} + v_{12}$

$L(v_{22}) = \lambda v_{22} + v_{21}$

$$m_J(t) = lcm\left((t-\lambda)^3, (t-\lambda)^2\right) = (t-\lambda)^3 \text{ and}$$

$$(J - \lambda I_5)^3 = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & & \\ \hline & 0 & & 0 & 1 \\ & & & 0 & 0 \end{array}\right]^3 = 0_5^5.$$

Furthermore: $L_\lambda^{(1)} = L_\lambda = \langle v_{11}, v_{21} \rangle$ 2 dim'l

$L_\lambda^{(2)} = \langle v_{11}, v_{12}, v_{21}, v_{22} \rangle$ is 4 dim'l

$L_\lambda^{(3)} = \langle v_{11}, v_{12}, v_{13}, v_{21}, v_{22} \rangle$ is 5 dim'l, $L_\lambda^{(3)} = V$.

<u>Questions</u>: ① If $L: V \to V$, $\dim(V) = n$, L has only one e-value λ, so $\mathrm{Char}_L(t) = (t-\lambda)^n$, what are all the possible J-block matrices that could represent L?

② If you also knew $m_L(t) = (t-\lambda)^m$ for some $1 \le m \le n$, how would that affect the answer?

Answers: ① The blocks have sizes $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$ and $n_1 + n_2 + \cdots + n_r = n$ so the number of such choices is $p(n)$, the "partition" function which counts the number of ways to express $n \geq 1$ as a sum of positive integers. This is an important function in combinatorics with connections to number theory and physics.

Partitions: $1 = 1$ is the only one so $p(1) = 1$

$2 = 2 = 1+1$ so $p(2) = 2$

$3 = 3 = 2+1 = 1+1+1$ so $p(3) = 3$

$4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$ so $p(4) = 5$

$5 = 5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$, $p(5) = 7$

It is useful to define $p(0)=1$ and $\underline{(244}$

then to form the power series generating function $\displaystyle\sum_{n\geq 0} p(n)q^n = 1+1q+2q^2+3q^3+$

$5q^4+7q^5+\cdots$.

Note: $\dfrac{1}{1-q^i} = \displaystyle\sum_{m\geq 0} q^{mi}$ as a formal geom. ser.

$$\left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\cdots\left(\frac{1}{1-q^k}\right) = \left(\sum_{n_1\geq 0} q^{n_1 1}\right)\left(\sum_{n_2\geq 0} q^{n_2 2}\right)\cdots\left(\sum_{n_k\geq 0} q^{n_k k}\right)$$

$$= \sum_{n_1,\dots,n_k\geq 0} q^{n_1 1+n_2 2+\cdots+n_k k} = \sum_{n\geq 0} p_k(n)q^n$$ where $p_k(n)$ is the number of partitions of n into parts $\leq k$

Th: $\sum_{m\geq 0} p(m) q^m = \dfrac{1}{\prod_{n\geq 1}(1-q^n)} = \dfrac{1}{\phi(q)}$ $\underline{L245}$

Euler found a formula for $\phi(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n(3n+1)}$, which implies a recursion for values of $p(m)$.

Ramanujan studied congruences satisfied by values of $p(m)$, asymptotic formulas, etc.

This is a very interesting topic!

Think about Question 2 for later discussion.

Th: For $L: V \to V$, $\dim(V) = n$, if
$\mathrm{Char}_L(t) = \prod_{i=1}^{r} (t - \lambda_i)^{k_i}$, $m_L(t) = \prod_{i=1}^{r} (t - \lambda_i)^{m_i}$,
there is a basis T of V s.t. ${}_T[L]_T = J$
$= \mathrm{diag}(J_1, J_2, \ldots, J_r)$ and for $1 \le i \le r$
$J_i = \mathrm{diag}(J(\lambda_i, n_{i1}), J(\lambda_i, n_{i2}), \ldots, J(\lambda_i, n_{ig_i}))$
where $g_i = \dim(L_{\lambda_i})$, $n_{i1} \ge n_{i2} \ge \cdots \ge n_{ig_i} \ge 1$ and
$n_{i1} + n_{i2} + \cdots + n_{ig_i} = k_i$ and $n_{i1} = m_i$.

This matrix, J, is called the Jordan Canonical Form matrix of L.

Pf. We have already proved that $V = W_1 \oplus \cdots \oplus W_r$ where $W_i = \operatorname{Ker}((L - \lambda_i I_V)^{m_i})$ are L-invariant subspaces. If $L_i = L|_{W_i}$ we also know $\operatorname{Char}_{L_i}(t) = (t - \lambda_i)^{k_i}$ and $m_{L_i}(t) = (t - \lambda_i)^{m_i}$ for $1 \le i \le r$. So it is enough to prove the results for one e-value at a time, that is, for $r = 1$, so we can drop the subscript i and say there is one e-value λ, $\operatorname{Char}_L(t) = (t - \lambda)^k$, $m_L(t) = (t - \lambda)^m$.

If there is a basis T of V s.t. ${}_T[L]_T = J = \operatorname{diag}(J(\lambda, n_1), J(\lambda, n_2), \ldots, J(\lambda, n_g))$ then the order of the vectors in T can certainly be chosen so that $n_1 \ge n_2 \ge \cdots \ge n_g \ge 1$ and since each block has first column corresponding to an e-vector for L, the number of blocks is the geometric mult. $g = \dim(L_\lambda)$. $V = W_1$ means $k = n$, so

$n_1 + n_2 + \cdots + n_g = k = n$, and the min. poly.
$m_L(t) = \text{l.c.m.}(m_{J(\lambda,n_1)}(t), \ldots, m_{J(\lambda,n_g)}(t))$
$= \text{l.c.m.}((t-\lambda)^{n_1}, (t-\lambda)^{n_2}, \ldots, (t-\lambda)^{n_g}) = (t-\lambda)^{n_1}$ since
$n_1 \geq n_2 \geq \cdots \geq n_g \geq 1$, so $n_1 = m$.
We have already discussed this situation and used notation $L_\lambda^{(i)} = \ker((L-\lambda I_V)^i)$, $1 \leq i \in \mathbb{Z}$, and we know $L_\lambda^{(m)} = L_\lambda^{(m+1)} = L_\lambda^{(m+2)} = \cdots$ but for $1 \leq i < m$, $L_\lambda^{(i)} \subsetneqq L_\lambda^{(i+1)}$. As discussed on page 238, letting $v_m \in L_\lambda^{(m)} - L_\lambda^{(m-1)}$ and for $1 \leq i \leq m-1$, $v_{m-i} = (L-\lambda I_V)^i v_m$ gives $v_{m-1}, \ldots, v_1$ such that $T_1 = \{v_1, \ldots, v_m\}$ is indep. since $v_{m-i} \in L_\lambda^{(m-i)} - L_\lambda^{(m-i-1)}$.
If $W_1 = \langle T_1 \rangle$ then it is an L-invar. subspace and